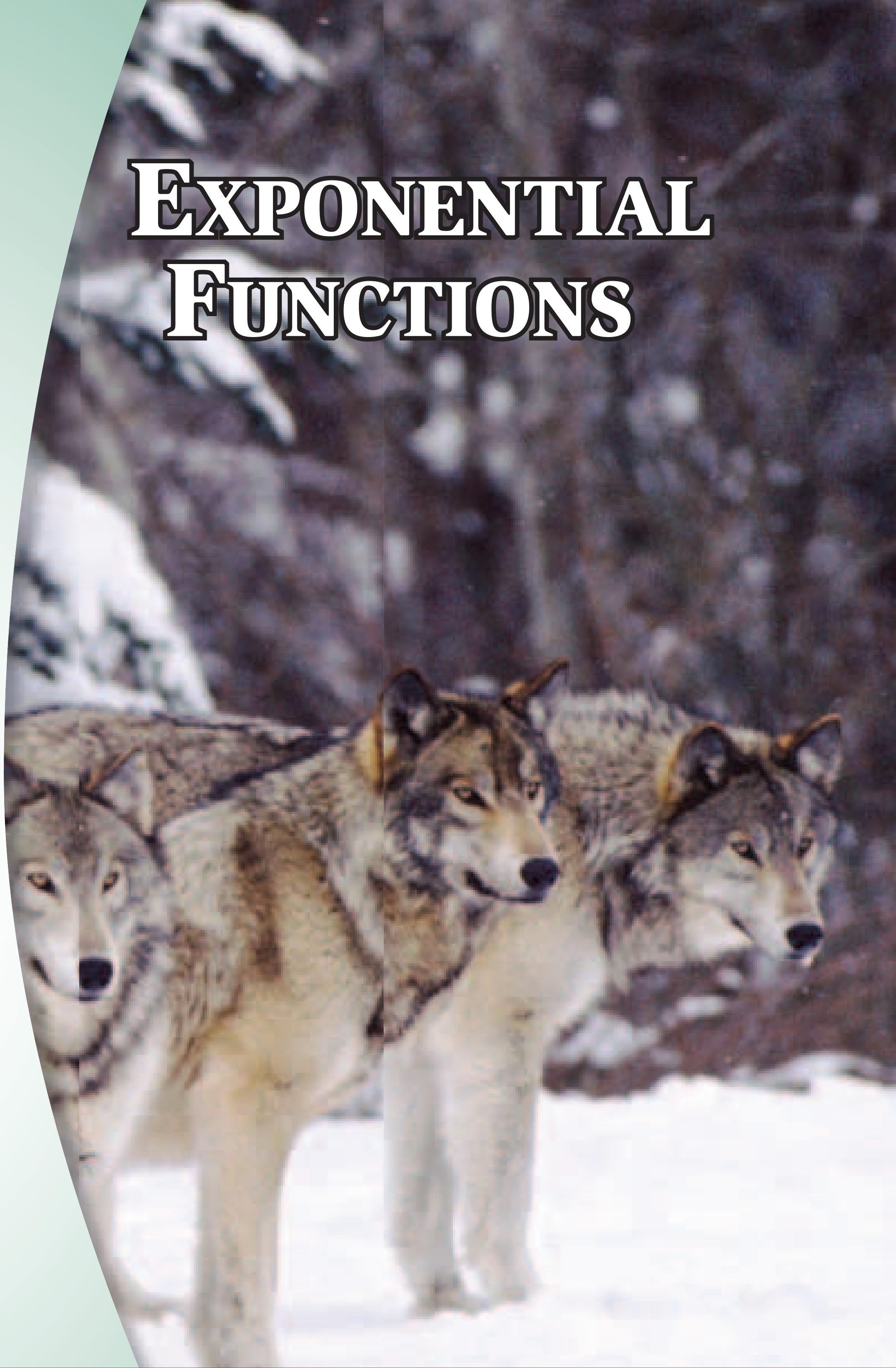
UNIT 5

EXPONENTIAL FUNCTIONS



In everyday conversation, the phrase "growing exponentially" is used to describe any situation where some quantity is increasing rapidly with the passage of time. But in mathematics, the terms *exponential growth* and *exponential decay* refer to particular important patterns of change.

For example, when wildlife biologists estimated the population of gray wolves in Michigan, Wisconsin, and Minnesota, they found it growing exponentially—at an annual rate of about 25% from a base of about 170 wolves in 1990 to about 3,100 wolves in 2003.

In this unit, you will develop understanding and skill required to study patterns of change like growth of the midwestern gray wolf population and decay of medicines in the human body.

The key ideas and strategies for studying those patterns will be developed in two lessons.

Lessons

1 *Exponential Growth*

Recognize situations in which variables grow exponentially over time. Write *NOW-NEXT* and "$y = \ldots$" rules that express those patterns of change. Use tables, graphs, and spreadsheets to solve problems related to exponential growth. Use properties of integer exponents to write exponential expressions in useful equivalent forms.

2 *Exponential Decay*

Recognize and solve problems in situations where variables decline exponentially over time. Use properties of fractional exponents to write exponential expressions in useful equivalent forms.

Tom Brakefield/CORBIS

LESSON 1

Exponential Growth

In the popular book and movie *Pay It Forward*, 12-year-old Trevor McKinney gets a challenging assignment from his social studies teacher.

Think of an idea for world change, and put it into practice!

Trevor came up with an idea that fascinated his mother, his teacher, and his classmates.

He suggested that he would do something really good for three people. Then when they ask how they can pay him back for the good deeds, he would tell them to “pay it forward”—each doing something good for three other people.

Trevor figured that those three people would do something good for a total of nine others. Those nine would do something good for 27 others, and so on. He was sure that before long there would be good things happening to billions of people all around the world.

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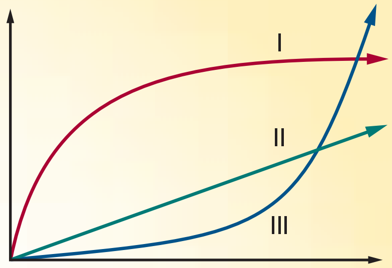
Think About This Situation

Continue Trevor's kind of Pay It Forward thinking.



a How many people would receive a Pay It Forward good deed at each of the next several stages of the process?

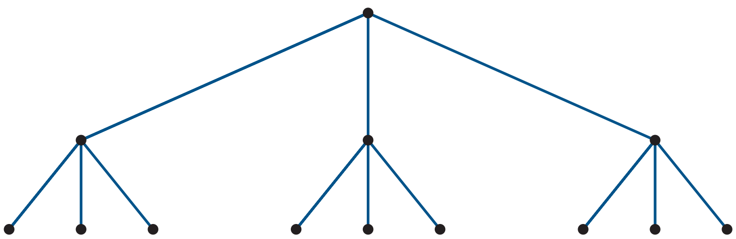
b What is your best guess about the number of people who would receive Pay It Forward good deeds at the tenth stage of the process?

c Which of the graphs at the right do you think is most likely to represent the pattern by which the number of people receiving Pay It Forward good deeds increases as the process continues over time?

In this lesson, you will discover answers to questions like these and find strategies for analyzing patterns of change called *exponential growth*. You will also discover some basic properties of exponents that allow you to write exponential expressions in useful equivalent forms.

Investigation 1 Counting in Tree Graphs

The number of good deeds in the Pay It Forward pattern can be represented by a *tree graph* that starts like this:



The vertices represent the people who receive and do good deeds. Each edge represents a good deed done by one person for another. As you work on the problems of this investigation, look for answers to these questions:

What are the basic patterns of exponential growth in variations of the Pay It Forward process?

How can those patterns be expressed with symbolic rules?

1 At the start of the Pay It Forward process, only one person does good deeds—for three new people. In the next stage, the three new people each do good things for three more new people. In the next stage, nine people each do good things for three more new people, and so on, with no person receiving more than one good deed.

a. Make a table that shows the number of people who will receive good deeds at each of the next seven stages of the Pay It Forward process. Then plot the (*stage, number of good deeds*) data.

Stage of Process	1	2	3	4	5	6	7	8	9	10
Number of Good Deeds	3	9	27							

b. How does the number of good deeds at each stage grow as the tree progresses? How is that pattern of change shown in the plot of the data?

c. How many stages of the Pay It Forward process will be needed before a total of at least 25,000 good deeds will be done?

2 Consider now how the number of good deeds would grow if each person touched by the Pay It Forward process were to do good deeds for only two other new people, instead of three.

a. Make a tree graph for several stages of this Pay It Forward process.

b. Make a table showing the number of good deeds done at each of the first 10 stages of the process and plot those sample (*stage, number of good deeds*) values.

c. How does the number of good deeds increase as the Pay It Forward process progresses in stages? How is that pattern of change shown in the plot of the data?

d. How many stages of this process will be needed before a total of 25,000 good deeds will have been done?

3 In the two versions of Pay It Forward that you have studied, you can use the number of good deeds at one stage to calculate the number at the next stage.

a. Use the words *NOW* and *NEXT* to write rules that express the two patterns.

b. How do the numbers and calculations indicated in the rules express the patterns of change in tables of (*stage, number of good deeds*) data?

c. Write a rule relating *NOW* and *NEXT* that could be used to model a Pay It Forward process in which each person does good deeds for four other new people. What pattern of change would you expect to see in a table of (*stage, number of good deeds*) data for this Pay It Forward process?

4 What are the main steps (not keystrokes) required to use a calculator to produce tables of values like those you made in Problems 1 and 2?

5 It is also convenient to have rules that will give the number of good deeds N at any stage x of the Pay It Forward process, without finding all the numbers along the way to stage x. When students in one class were given the task of finding such a rule for the process in which each person does three good deeds for others, they came up with four different ideas:

$$N = 3x$$
$$N = x + 3$$
$$N = 3^x$$
$$N = 3x + 1$$

a. Are any of these rules for predicting the number of good deeds N correct? How do you know?

b. How can you be sure that the numbers and calculations expressed in the correct "$N = \ldots$" rule will produce the same results as the *NOW-NEXT* rule you developed in Problem 3?

c. Write an "$N = \ldots$" rule that would show the number of good deeds at stage number x if each person in the process does good deeds for two others.

d. Write an "$N = \ldots$" rule that gives the number of good deeds at stage x if each person in the process does good deeds for four others.

Summarize the Mathematics

Look back at the patterns of change in the number of good deeds in the different Pay It Forward schemes—three per person and two per person.

a Compare the processes by noting similarities and differences in:

i. Patterns of change in the tables of (*stage*, *number of good deeds*) data;

ii. Patterns in the graphs of (*stage*, *number of good deeds*) data;

iii. The rules relating *NOW* and *NEXT* numbers of good deeds; and

iv. The rules expressing number of good deeds N as a function of stage number x.

b Compare patterns of change in numbers of good deeds at each stage of the Pay It Forward process to those of linear functions that you have studied in earlier work.

i. How are the *NOW-NEXT* rules similar, and how are they different?

ii. How are the "$y = \ldots$" rules similar, and how are they different?

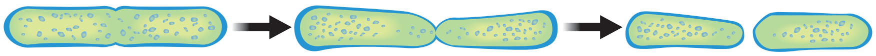
iii. How are the patterns of change in tables and graphs of linear functions similar to those of the Pay It Forward examples, and how are they different?

Be prepared to share your ideas with the rest of the class.

Check Your Understanding

The patterns in spread of good deeds by the Pay It Forward process occur in other quite different situations. For example, when bacteria infect some part of your body, they often grow and split into pairs of genetically equivalent cells over and over again.

a. Suppose a single bacterium lands in a cut on your hand. It begins spreading an infection by growing and splitting into two bacteria every 20 minutes.



i. Complete a table showing the number of bacteria after each 20-minute period in the first three hours. (Assume none of the bacteria are killed by white blood cells.)

Number of 20-min Periods	1	2	3
Bacteria Count	2	4	

ii. Plot the (*number of time periods, bacteria count*) values.

iii. Describe the pattern of growth of bacteria causing the infection.

b. Use *NOW* and *NEXT* to write a rule relating the number of bacteria at one time to the number 20 minutes later. Then use the rule to find the number of bacteria after fifteen 20-minute periods.

c. Write a rule showing how the number of bacteria N can be calculated from the number of stages x in the growth and division process.

d. How are the table, graph, and symbolic rules describing bacteria growth similar to and different from the Pay It Forward examples? How are they similar to, and different from, typical patterns of linear functions?

Investigation 2 Getting Started

The patterns of change that occur in counting the good deeds of a Pay It Forward scheme and the growing number of bacteria in a cut are examples of *exponential growth*. Exponential functions get their name from the fact that in rules like $N = 2^x$ and $N = 3^x$, the independent variable occurs as an exponent. As you work on the problems in this investigation, look for answers to the following questions:

What are the forms of NOW-NEXT and "y = …" rules for basic exponential functions?

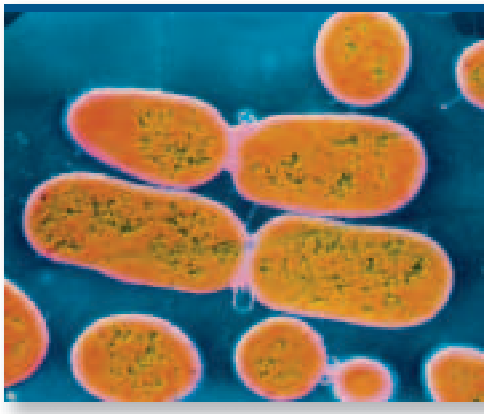
How can those rules be modified to model other similar patterns of change?

1. Infections seldom start with a single bacterium. Suppose that you cut yourself on a rusty nail that puts 25 bacteria cells into the wound. Suppose also that those bacteria divide in two after every quarter of an hour.

   a. Make and record a guess of how many bacteria you think would be present in the cut after 8 hours (32 quarter-hours) if the infection continues to spread as predicted. (Assume that your body does not fight off the infection and you do not apply medication.) Then answer the following questions to check your ability to estimate the rate of exponential growth.

   b. Complete a table showing the first several numbers in the bacteria growth pattern:

Number of Quarter-Hour Periods	0	1	2	3	4
Number of Bacteria in the Cut	25	50			



   c. Use *NOW* and *NEXT* to write a rule showing how the number of bacteria changes from one quarter-hour to the next, starting from 25 at time 0.

   d. Write a rule showing how to calculate the number of bacteria N in the cut after x quarter-hour time periods.

   e. Use the rules in Parts c and d to calculate the number of bacteria after 8 hours. Then compare the results to each other and to your initial estimate in Part a.

2. Compare the pattern of change in this situation to the simple case that started from a single bacterium by noting similarities and differences in the:

   a. tables of (*number of time periods, bacteria count*) values;

   b. graphs of (*number of time periods, bacteria count*) values; and

   c. *NOW-NEXT* and "N = ..." rules.

3. Investigate the number of bacteria expected after 8 hours if the starting number of bacteria is 30, 40, 60, or 100, instead of 25. For each starting number at time 0, complete Parts a–c. (Divide the work among your classmates.)

   a. Make a table of (*number of time periods, bacteria count*) values for 8 quarter-hour time periods.

   b. Write two rules that model the bacteria growth—one relating *NOW* and *NEXT* and the other beginning "N = ..."

   c. Use each rule to find the number of bacteria after 8 hours and check that you get the same results.

   d. Now compare results from two of the cases—starting at 30 and starting at 40.

      i. How are the *NOW-NEXT* and "N = ..." rules for bacteria counts similar, and how are they different?

      ii. How are patterns in the tables and graphs of (*number of time periods, bacteria count*) data similar, and how are they different?

Just as bacteria growth won't always start with a single cell, other exponential growth processes can start with different initial numbers. Think again about the Pay It Forward scheme in Investigation 1.

4 Suppose that four good friends decide to start their own Pay It Forward tree. To start the tree, they each do good deeds for three different people. Each of those new people in the tree does good deeds for three other new people, and so on.

a. What *NOW-NEXT* rule shows how to calculate the number of good deeds done at each stage of this tree?

b. What "$N = \ldots$" rule shows how to calculate the number of good deeds done at any stage x of this tree?

c. How would the *NOW-NEXT* and "$N = \ldots$" rules be different if the group of friends starting the tree had five members instead of four?

d. Which of the Pay It Forward schemes below would most quickly reach a stage in which 1,000 good deeds are done? Why does that make sense?

Scheme 1: Start with a group of four friends and have each person in the tree do good deeds for two different people; or

Scheme 2: Start with only two friends and have each person in the tree do good deeds for three other new people.

In studying exponential growth, it is helpful to know the *initial value* of the growing quantity. For example, the initial value of the growing bacteria population in Problem 1 was 25. You also need to know when the initial value occurs. For example, the bacteria population was 25 after 0 quarter-hour periods.

In Problem 4 on the other hand, 12 good deeds are done at Stage 1. In this context, "Stage 0" does not make much sense, but we can extend the pattern backward to reason that $N = 4$ when $x = 0$.

5 Use your calculator and the [^] key to find each of the following values: 2^0, 3^0, 5^0, 23^0.

a. What seems to be the calculator value for b^0, for any positive value of b?

b. Recall the examples of exponential patterns in bacterial growth. How do the "$N = \ldots$" rules for those situations make the calculator output for b^0 reasonable?

6 Now use your calculator to make tables of (x, y) values for each of the following functions. Use integer values for x from 0 to 6. Make notes of your observations and discussion of questions in Parts a and b.

i. $y = 5(2^x)$ **ii.** $y = 4(3^x)$

iii. $y = 3(5^x)$ **iv.** $y = 7(2.5^x)$

a. What patterns do you see in the tables? How do the patterns depend on the numbers in the function rule?

b. What differences would you expect to see in tables of values and graphs of the two exponential functions $y = 3(6^x)$ and $y = 6(3^x)$?

7 Suppose you are on a team studying the growth of bacteria in a laboratory experiment. At the start of your work shift in the lab, there are 64 bacteria in one petri dish culture, and the population seems to be doubling every hour.

a. What rule should predict the number of bacteria in the culture at a time x hours after the start of your work shift?

b. What would it mean to calculate values of y for negative values of x in this situation?

c. What value of y would you expect for $x = -1$? For $x = -2$? For $x = -3$ and -4?

d. Use your calculator to examine a table of (x, y) values for the function $y = 64(2^x)$ when $x = 0, -1, -2, -3, -4, -5, -6$. Compare results to your expectations in Part c. Then explain how you could think about this problem of bacteria growth in a way so that the calculator results make sense.

8 Study tables and graphs of (x, y) values to estimate solutions for each of the following equations and inequalities. In each case, be prepared to explain what the solution tells about bacteria growth in the experiment of Problem 7.

a. $1{,}024 = 64(2^x)$

b. $8{,}192 = 64(2^x)$

c. $64(2^x) > 25{,}000$

d. $4 = 64(2^x)$

e. $64(2^x) < 5{,}000$

f. $64(2^x) = 32$

Summarize the Mathematics

The exponential functions that you studied in this investigation describe patterns of change in bacteria growth and numbers of people in a Pay It Forward tree. They have some features in common.

a Each *NOW-NEXT* rule fits the pattern *NEXT* = b • *NOW*, starting at a. What do the values of b and a tell about the pattern of change represented by the *NOW-NEXT* rule? How will that pattern be illustrated in a table or a graph of (x, y) values?

b Each "$y = \ldots$" rule fits the pattern $y = a(b^x)$. What do the values of a and b tell about the pattern of change represented by the rule? How will that pattern be illustrated in a table or a graph of (x, y) values?

c What is the value of b^x, when x is 0? What would this result mean in a problem situation where exponential growth is being studied?

d How would you calculate values of b^x when x is a negative number? What would those results mean in a problem situation where exponential growth is being studied?

Be prepared to explain your ideas to the entire class.

Scimat/Photo Researchers

Check Your Understanding

Alexander Fleming
Discoverer of Penicillin

The drug penicillin was discovered by observation of mold growing on biology lab dishes. Suppose a mold begins growing on a lab dish. When first observed, the mold covers 7 cm^2 of the dish surface, but it appears to double in area every day.

a. What rules can be used to predict the area of the mold patch 4 days after the first measurement:

- **i.** using *NOW-NEXT* form?
- **ii.** using "$y = \ldots$" form?

b. How would each rule in Part a change if the initial mold area was only 3 cm^2?

c. How would each rule in Part a change if the area of the mold patch increased by a factor of 1.5 every day?

d. What mold area would be predicted after 5 days in each set of conditions from Parts a–c?

e. For "$y = \ldots$" rules used in calculating growth of mold area, what would it mean to calculate values of y when x is a negative number?

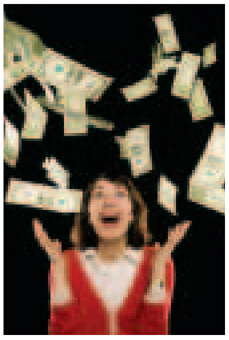
f. Write and solve equations or inequalities that help to answer these questions.

- **i.** If the area of a mold patch is first measured to be 5 cm^2 and the area doubles each day, how long will it take that mold sample to grow to an area of 40 cm^2?
- **ii.** For how many days will the mold patch in part i have an area less than 330 cm^2?

Investigation 3 Compound Interest

Every now and then you may hear about somebody winning a big payoff in a state lottery. The winnings can be 1, 2, 5, or even 100 million dollars. The big money wins are usually paid off in annual installments for about 20 years. But some smaller prizes are paid at once. How would you react if this news report were actually about you?

Kalamazoo Teen Wins Big Lottery Prize



A Kalamazoo teenager has just won the daily lottery from a Michigan lottery ticket that she got as a birthday gift from her uncle. In a new lottery payoff scheme, the teen has two payoff choices.

One option is to receive a single $10,000 payment now.

In the other plan, the lottery promises a single payment of $20,000 ten years from now.

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1 Imagine that you had just won that Michigan lottery prize.

a. Discuss with others your thinking on which of the two payoff methods to choose.

b. Suppose a local bank called and said you could invest your $10,000 payment in a special 10-year certificate of deposit (CD), earning 8% interest compounded yearly. How would this affect your choice of payoff method?

As you work on the problems of this investigation, look for answers to the question:

How can you represent and reason about functions involved in investments paying compound interest?

Of the two lottery payoff methods, one has a value of $20,000 at the end of 10 years. The value (in 10 years) of receiving the $10,000 payoff now and putting it in a 10-year certificate of deposit paying 8% interest compounded annually is not so obvious.

- After one year, your balance will be:
  $10{,}000 + (0.08 \times 10{,}000) = 1.08 \times 10{,}000 = \$10{,}800$.
- After the second year, your balance will be:
  $10{,}800 + (0.08 \times 10{,}800) = 1.08 \times 10{,}800 = \$11{,}664$.

During the next year, the CD balance will increase in the same way, starting from $11,664, and so on.

2 Write rules that will allow you to calculate the balance of this certificate of deposit:

a. for the next year, using the balance from the current year.

b. after any number of years x.

3 Use the rules from Problem 2 to determine the value of the certificate of deposit after 10 years. Then decide which 10-year plan will result in more money and how much more money that plan will provide.

4 Look for an explanation of your conclusion in Problem 3 by answering these questions about the potential value of the CD paying 8% interest compounded yearly.

a. Describe the pattern of growth in the CD balance as time passes.

b. Why isn't the change in the CD balance the same each year?

c. How is the pattern of increase in CD balance shown in the shape of a graph for the function relating CD balance to time?

d. How could the pattern of increase have been predicted by thinking about the rules (*NOW-NEXT* and "$y = \ldots$") relating CD balance to time?

5 Suppose that the prize winner decided to leave the money in the CD, earning 8% interest for more than 10 years. Use tables or graphs to estimate solutions for the following equations and inequalities. In each case, be prepared to explain what the solution tells about the growth of a $10,000 investment that earns 8% interest compounded annually.

a. $10{,}000(1.08^x) = 25{,}000$

b. $10{,}000(1.08^x) = 37{,}000$

c. $10{,}000(1.08^x) = 50{,}000$

d. $10{,}000(1.08^x) \geq 25{,}000$

e. $10{,}000(1.08^x) \leq 30{,}000$

f. $10{,}000(1.08^x) = 10{,}000$

6 Compare the pattern of change and the final account balance for the plan that invests $10,000 in a CD that earns 8% interest compounded annually over 10 years to those for the following possible savings plans over 10 years. Write a summary of your findings.

a. Initial investment of $15,000 earning only 4% annual interest compounded yearly

b. Initial investment of $5,000 earning 12% annual interest compounded yearly

Summarize the Mathematics

Most savings accounts operate in a manner similar to the bank's certificate of deposit offer. However, they may have different starting balances, different interest rates, or different periods of investment.

a Describe two ways to find the value of such a savings account at the end of each year from the start to year 10. Use methods based on:

i. a rule relating *NOW* and *NEXT*.

ii. a rule like $y = a(b^x)$.

b What graph patterns would you expect from plots of (*year*, *account balance*) values?

c How would the function rules change if the interest rate changes? If the initial investment changes?

d Why does the dollar increase in the account balance get larger from each year to the next?

e How are the patterns of change that occur with the bank investment similar to and different from those of other functions that you've used while working on problems of Investigations 1 and 2? On problems of previous units?

Be prepared to explain your methods and ideas to the entire class.

Check Your Understanding

In solving change-over-time problems in Unit 1, you discovered that the world population and populations of individual countries grow in much the same pattern as money earning interest in a bank. For example, you used data like the following to predict population growth in two countries.

- Brazil is the most populous country in South America. In 2005, its population was about 186 million. It was growing at a rate of about 1.1% per year.
- Nigeria is the most populous country in Africa. Its 2005 population was about 129 million. It was growing at a rate of about 2.4% per year.

a. Assuming that these growth rates continue, write function rules to predict the populations of these countries for any number of years x in the future.

b. Compare the patterns of growth expected in each country for the next 20 years. Use tables and graphs of (*year since 2005, population*) values to illustrate the similarities and differences you notice.

c. Write and solve equations that give estimates when:

i. Brazil's population might reach 300 million.

ii. Nigeria's population might reach 200 million.

d. Assuming these growth patterns continue, estimate when the population of Nigeria will be greater than the population of Brazil.

Investigation 4 Modeling Data Patterns

In the *Patterns of Change* unit, you used data about wildlife populations to make predictions and to explore effects of protection and hunting policies. For example, you used information from studies of Midwest wolf populations to predict growth over time in that species. You used information about Alaskan bowhead whale populations and hunting rates to make similar projections into the future.

In each case, you began the prediction with information about the current populations and the growth rates as percents. It's not hard to imagine how field biologists might count wolves or whales by patient observation. But they can't observe percent growth rates, and those rates are unlikely to be constant from one year to the next. As you work on the problems of this investigation, look for answers to the following question:

What are some useful strategies for finding functions modeling patterns of change that are only approximately exponential?

1. Suppose that census counts of Midwest wolves began in 1990 and produced these estimates for several different years:

Time Since 1990 (in years)	0	2	5	7	10	13
Estimated Wolf Population	100	300	500	900	1,500	3,100

a. Plot the wolf population data and decide whether a linear or exponential function seems likely to match the pattern of growth well. For the function type of your choice, experiment with different rules to see which rule provides a good model of the growth pattern.

b. Use your calculator or computer software to find both linear and exponential regression models for the given data pattern. Compare the fit of each function to the function you developed by experimentation in Part a.

c. What do the numbers in the linear and exponential function rules from Part b suggest about the pattern of change in the wolf population?

d. Use the model for wolf population growth that you believe to be best to calculate population estimates for the missing years 1994 and 2001 and then for the years 2015 and 2020.

2. Suppose that census counts of Alaskan bowhead whales began in 1970 and produced these estimates for several different years:

Time Since 1970 (in years)	0	5	15	20	26	31
Estimated Whale Population	5,040	5,800	7,900	9,000	11,000	12,600

a. Plot the given whale population data and decide which type of function seems likely to match the pattern of growth well. For the function type of your choice, experiment with different rules to see which provides a good model of the growth pattern.

b. Use your calculator or computer software to find both linear and exponential regression models for the data pattern. Compare the fit of each function to that of the function you developed by experimentation in Part a.

c. What do the numbers in the linear and exponential function rules from Part b suggest about patterns of change in the whale population?

d. Use the model for whale population growth that you believe to be best to calculate population estimates for the years 2002, 2005, and 2010.

Summarize the Mathematics

In the problems of this investigation, you studied ways of finding function models for growth patterns that could only be approximated by one of the familiar types of functions.

a How do you decide whether a data pattern is modeled best by a linear or an exponential function?

b What do the numbers *a* and *b* in a linear function $y = a + bx$ tell about patterns in:

i. the graph of the function?

ii. a table of (x, y) values for the function?

c What do the numbers *c* and *d* in an exponential function $y = c(d^x)$ tell about patterns in:

i. the graph of the function?

ii. a table of (x, y) values for the function?

d What strategies are available for finding a linear or exponential function that models a linear or exponential data pattern?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Test your ideas about the connections between functions, problem conditions, and data patterns.

a. What *NOW-NEXT* and "$y = \ldots$" rules will express patterns of change in which a variable quantity is increasing:

i. at a rate of 20% per year from a starting value of 750?

ii. at a rate of 4.5% per month from a starting value of 35?

iii. at a rate of 24 per day from a starting value of 18?

b. Write functions that provide good models for the patterns of change that relate *p*, *q*, and *r* to *x* in the following tables.

i.

x	−10	−5	0	6	15	20	30
p	1	3	5	8	12	15	18

ii.

x	−10	−5	0	6	15	20	30
q	1	8	60	650	25,000	190,000	11,000,000

iii.

x	−10	−5	0	6	15	20	30
r	1.0	1.3	1.6	2.25	3.4	4.4	7.0

Investigation 5 Properties of Exponents I

In solving the problems in Investigations 1–4, you focused on functions modeling exponential growth. You used what you knew about the problem situations to guide development of the function models, to plan calculations that would answer the given questions, and to interpret information in rules, tables, and graphs. For example, you developed and used the rules $B = 2^x$ and $B = 25(2^x)$ to study the pattern of bacteria growth in a cut.

Since doubling occurs so often in questions about exponential growth, it is helpful to know some basic powers of 2, like $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, … , $2^9 = 512$, and $2^{10} = 1{,}024$. When students in one Wisconsin class had memorized those facts, someone suggested reasoning about even higher powers like this:

Since 2^{10} is about 1,000: the value of 2^{11} should be about 2,000,
the value of 2^{12} should be about 4,000,
the value of 2^{13} should be about 8,000,
⋮
the value of 2^{20} should be about 1,000,000.

How do you suppose the student was thinking about exponents to come up with that estimation strategy?

To develop and test strategies for working with exponential expressions, it helps to know some basic methods of writing these expressions in useful equivalent forms. Remember that the starting point in work with exponents is an expression like b^n, where b is any real number and n is any non-negative integer. The number b is called the **base** of the exponential expression, and n is called the **exponent** or the **power**.

$$b^n = b \cdot b \cdot b \cdot \cdots \cdot b \text{ (n factors)} \quad \text{and} \quad b^0 = 1 \text{ (for } b \neq 0)$$

As you work on the problems in this investigation, look for answers to this question:

How can the above definition of exponent be used to discover and justify other properties of exponents that make useful algebraic manipulations possible?

Products of Powers Work with exponents is often helped by writing products like $b^x \cdot b^y$ in simpler form or by breaking a calculation like b^z into a product of two smaller numbers.

1 Find values for w, x, and y that will make these equations true statements:

a. $2^{10} \cdot 2^3 = 2^y$

b. $5^2 \cdot 5^4 = 5^y$

c. $3 \cdot 3^7 = 3^y$

d. $2^w \cdot 2^4 = 2^7$

e. $b^4 \cdot b^2 = b^y$

f. $9^w \cdot 9^x = 9^5$

2 Examine the results of your work on Problem 1.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

3 When people work with algebraic expressions that involve exponents, there are some common errors that slip into the calculations. How would you help other students correct their understanding of operations with exponents if their work showed the following errors?

a. $3^5 = 15$

b. $3^4 \cdot 5^2 = 15^8$

c. $3^4 \cdot 5^2 = 15^6$

d. $3^4 + 3^2 = 3^6$

Power of a Power You know that $8 = 2^3$ and $64 = 8^2$, so $64 = (2^3)^2$. As you work on the next problems, look for a pattern suggesting how to write equivalent forms for expressions like $(b^x)^y$ that involve powers of powers.

4 Find values for x and z that will make these equations true statements:

a. $(2^3)^4 = 2^z$

b. $(3^5)^2 = 3^z$

c. $(5^2)^x = 5^6$

d. $(b^2)^5 = b^z$

5 Examine the results of your work on Problem 4.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as common errors in evaluating a power of a power like $(4^3)^2$? How would you help someone who made those errors correct their understanding of how exponents work?

Power of a Product The area of a circle can be calculated from its radius r using the formula $A = \pi r^2$. It can be calculated from the diameter using the formula $A = \pi(0.5d)^2$. Next search for a pattern showing how powers of products like $(0.5d)^2$ can be expressed in equivalent forms.

6 Find values for x and y that will make these equations true statements:

a. $(6 \cdot 11)^3 = 6^x \cdot 11^y$

b. $(3\pi)^4 = 3^x \cdot \pi^y$

c. $(2m)^3 = 2^x m^y$

d. $(m^3 p)^2 = m^x p^y$

7 Examine the results of your work on Problem 6.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating a power of a product like $(4t)^3$? How would you help someone who made those errors correct their understanding of how exponents work?

Summarize the Mathematics

The problems of this investigation asked you to formulate, test, and justify several principles that allow writing of exponential expressions in convenient equivalent forms.

a For each of the properties of exponents you explored in Problems 1–7, how would you explain the property in words that describe the relationship between two equivalent forms of exponential expressions?

b Summarize the properties of exponents you explored in Problems 1–7 by completing these statements to show equivalent forms for exponential expressions:

i. $b^m \cdot b^n = \ldots$ **ii.** $(b^m)^n = \ldots$ **iii.** $(ab)^n = \ldots$

c What examples would you use to illustrate common errors in use of exponents, and how would you explain the errors in each example?

Be prepared to share your explanations and reasoning with the class.

Check Your Understanding

Use properties of exponents to write each of the following expressions in another equivalent form. Be prepared to explain how you know your answers are correct.

a. $(y^3)(y^6)$

b. $(5x^2y^4)(2xy^3)$

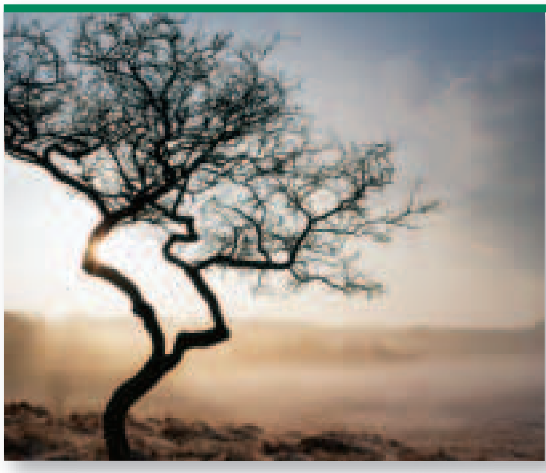
c. $(pq)^3$

d. $(p^3)^5$

e. $(7p^3q^2)^2$

On Your Own

Applications



1 Imagine a tree that each year grows 3 new branches from the end of each existing branch. Assuming that your tree is a single stem when it is planted:

a. How many new branches would you expect to appear in the first year of new growth? How about in the second year of new growth?

b. Write a rule that relates the number of new branches *B* to the year of growth *R*.

c. In what year will the number of new branches first be greater than 15,000?

2 The Silver Spring Soccer Club has boys and girls from about 750 families who play soccer each Saturday. When it is rainy, everyone wants to know if the games will be canceled. The club president makes a decision and then calls two families. Each of them calls two more different families. Each of those families calls two more different families, and so on.

a. Sketch a tree graph that shows how the number of people called grows in stages from the first calls by the club president. What do the vertices of the tree graph represent? What do the edges represent?

b. Make a table and a graph showing the number of calls made at each of the first 10 stages of this calling tree.

c. Write two rules that can be used to calculate the number of calls made at various stages of this calling tree—one in *NOW-NEXT* form and another in "$y = \ldots$" form.

d. How many stages of the calling tree will be needed before all 750 families are contacted?

3 The bacteria *E. coli* often cause illness among people who eat the infected food. Suppose a single *E. coli* bacterium in a batch of ground beef begins doubling every 10 minutes.

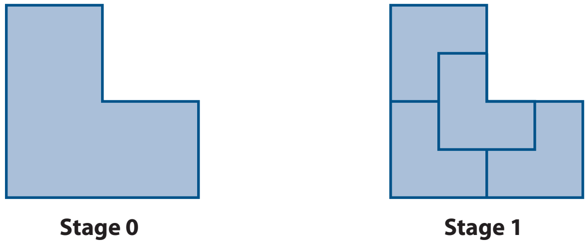
a. How many bacteria will there be after 10, 20, 30, 40, and 50 minutes have elapsed? Assume no bacteria die.

b. Write two rules that can be used to calculate the number of bacteria in the food after any number of 10-minute periods—one using *NOW* and *NEXT*, and another beginning "$y = \ldots$."

c. Use your rules to make a table showing the number of *E. coli* bacteria in the batch of ground beef at the end of each 10-minute period over 2 hours. Then describe the pattern of change in the number of bacteria from each time period to the next.

d. Find the predicted number of bacteria after 4, 5, and 6 hours.

4 The left figure shown below is called a "chair." It can be subdivided into four congruent, smaller "chairs" as shown at the right. Each of the smaller chairs can be subdivided into four congruent, still smaller chairs, and this process can be continued.



a. Draw a picture of Stage 2 in the process that creates smaller "chairs" and count the number of small chairs at this stage.

b. Make a table that shows the number of small chairs at each stage of the process.

Stage	0	1	2	3	4	5	...	n
Number of "Chairs"	1	4						

c. Write a *NOW-NEXT* rule that shows how the number of chairs increases from each stage to the next.

5 Suppose that the Silver Spring Soccer Club has a meeting of the four club directors to decide on whether or not to cancel a scheduled game. Then the directors each start a branch of a calling tree by calling three families, and each of those families then calls three more families. This process continues until all 750 families are contacted.

a. Sketch a tree graph that shows how the number of people called grows in stages from the first calls by the club directors.

b. Make a table and a graph showing the number of calls made at each of the first 4 stages of this calling tree.

c. Write two rules that can be used to calculate the number of calls made at various stages of this calling tree—one in *NOW-NEXT* form and another in "$y = \ldots$" form.

d. How many stages of the calling tree will be needed before all 750 families are contacted?

6 Suppose 50 *E. coli* bacteria are introduced into some food as it's being processed, and the bacteria begin doubling every 10 minutes.

a. Make a table and a graph showing the number of bacteria from Stage 0 to Stage 6 of the infection process.

b. Write two rules that can be used to calculate the number of bacteria infecting the food at various stages of this process—one in *NOW-NEXT* form and another in "$y = \ldots$" form.

c. Predict the number of bacteria present after 3 hours. Explain how you made your prediction.

7 Suppose that a local benefactor wants to offer college scholarships to every child entering first grade at an elementary school in her community. For each student, the benefactor puts $5,000 in a separate savings fund that earns 5% interest compounded annually.

a. Make a table and a graph to show growth in the value of each account over the 12 years leading up to college entry.

b. Compare the pattern of growth of the account in Part a to one in which the initial deposit is $10,000. Compare values of each account after 12 years.

c. Compare the pattern of growth of the account in Part a to one in which the interest rate is 10% and the initial deposit is $5,000. Compare values of each account after 12 years.

d. Compare values of the accounts in Parts b and c after 12 years. What does this suggest about the relative importance of interest rate and initial balance in producing growth of an investment earning compound interest?

8 In 2000, the number of people worldwide living with HIV/AIDS was estimated at more than 36 million. That number was growing at an annual rate of about 15%.

a. Make a table showing the projected number of people around the world living with HIV/AIDS in each of the ten years after 2000, assuming the growth rate remains 15% per year.

b. Write two different kinds of rules that could be used to estimate the number of people living with HIV/AIDS at any time in the future.

c. Use the rules from Part b to estimate the number of people living with HIV/AIDS in 2015.

d. What factors might make the estimate of Part c an inaccurate forecast?

On Your Own

9 Studies in 2001 gave a low estimate of 7,700 for the population of Arctic bowhead whales. The natural annual growth rate was estimated to be about 3%. The harvest by Inuit people is very small in relation to the total population. Disregard the harvest for this task.

a. If the growth rate continued at 3%, what populations would be predicted for each year to 2010, using the low 2001 population estimate?

b. Which change of assumptions will lead to a greater 2010 whale population estimate

i. increasing the assumed population annual growth rate to 6%, with the 2001 low population estimate of 7,700, or

ii. increasing the 2001 population estimate to 14,400, but maintaining the 3% growth rate?

c. Find the time it takes for the whale population to double under each of the three sets of assumptions in Parts a and b.

10 The Dow Jones Industrial Average provides one measure of the "health" of the U.S. economy. It is a weighted average of the stock prices for 30 major American corporations. The following table shows the low point of the Dow Jones Industrial Average in selected years from 1965 to 2005.

Year	DJIA Low
1965	840
1970	631
1975	632
1980	759
1985	1,185
1990	2,365
1995	3,832
2000	9,796
2005	10,012

Source: www.analyzeindices.com/dow-jones-history.shtml

a. Find what you believe are the best possible linear and exponential models for the pattern of change in the low value of the Dow Jones Industrial Average over the time period shown in the table (use $t = 0$ to represent 1965). Then decide which you think is the better of the two models and explain your choice.

b. Use your chosen predictive model from Part a to estimate the low value of this stock market average in 2010 and 2015. Explain why you might or might not have confidence in those estimates.

c. Some stockbrokers who encourage people to invest in common stocks claim that one can expect an average return of 10% per year on that investment. Does the rule you chose to model increase in the Dow Jones average support that claim? Why or why not?

11 The following table shows the number of votes cast in a sample of U.S. Presidential elections between 1840 and 2004.

Year of Election	Major Party Candidates	Total Votes Cast
1840	Harrison vs. Van Buren	2,411,118
1860	Lincoln vs. Douglas	4,685,030
1880	Garfield vs. Hancock	9,218,951
1900	McKinley vs. Bryan	14,001,733
1920	Harding vs. Cox	26,757,946
1940	Roosevelt vs. Wilkie	49,752,978
1960	Kennedy vs. Nixon	68,836,385
1980	Reagan vs. Carter	86,515,221
2000	Bush vs. Gore	105,405,100
2004	Bush vs. Kerry	122,267,553

Source: en.wikipedia.org

a. Find rules for what you think are the best possible linear and exponential models of the trend relating votes cast to time (use $t = 0$ to represent the year 1840).

b. Which type of model—linear or exponential—seems to better fit the data pattern? Why do you think that choice is reasonable?

c. In what ways is neither the linear nor the exponential model a good fit for the data pattern relating presidential election votes to time? Why do you think that modeling problem occurs?

12 In 1958, Walter O'Malley paid about $700,000 to buy the Brooklyn Dodgers baseball team. He moved the team to Los Angeles, and in 1998 his son and daughter sold the team for $350,000,000. Assume that the team's value increased exponentially in annual increments according to a rule like $v = a(b^t)$, where $t = 0$ represents the year 1958.

a. What value of a is suggested by the given information?

b. Experiment to find a value of b that seems to give a rule that matches growth in team value prescribed by the given information.

c. What annual percent growth rate does your answer to Part b suggest for the value of the Dodgers team business?

d. According to the model derived in Parts a and b, when did the value of the Dodgers team first reach $1,000,000? $10,000,000? $100,000,000?

13 Find values for w, x, and y that will make these equations true statements.

a. $5^4 \cdot 5^5 = 5^y$ **b.** $3^6 \cdot 3^4 = 3^y$ **c.** $5^3 \cdot 5 = 5^y$

d. $7^w \cdot 7^6 = 7^{11}$ **e.** $1.5^w \cdot 1.5^x = 1.5^6$ **f.** $c^3 \cdot c^5 = c^y$

14 Write each of the following expressions in a simpler equivalent exponential form.

a. $7^4 \cdot 7^9$ **b.** $4.2^2 \cdot 4.2^5$ **c.** $x \cdot x^4$

d. $(c^2)(c^5)$ **e.** $(5x^3y^4)(4x^2y)$ **f.** $(7a^3bm^5)(b^4m^2)$

g. $(4x^3y^5)(10x)$ **h.** $(-2c^4d^2)(-cd)$

15 Find values for x, y, and z that will make these equations true statements.

a. $(7^5)^2 = 7^z$ **b.** $(4.5^2)^3 = x^6$ **c.** $(9^3)^x = 9^{12}$

d. $(t^3)^7 = t^z$ **e.** $(7 \cdot 5)^4 = 7^x \cdot 5^y$ **f.** $(3t)^4 = 3^x t^y$

g. $(5n^3)^2 = 5^x n^y$ **h.** $(c^5d^3)^2 = c^x d^y$

16 Write each of the following expressions in a simpler equivalent exponential form.

a. $(x^2)^3$ **b.** $(5a^3c^4)^2$ **c.** $(3xy^4z^2)^4$ **d.** $(-5x^3)^2$

Connections

17 Partially completed tables for four relations between variables are given below. In each case, decide if the table shows an exponential or a linear pattern of change. Based on that decision, complete a copy of the table as the pattern suggests. Then write rules for the patterns in two ways: using rules relating *NOW* and *NEXT* y values and using rules beginning "$y = \ldots$" for any given x value.

a.

x	0	1	2	3	4	5	6	7	8
y				8	16	32			

b.

x	0	1	2	3	4	5	6	7	8
y				40	80	160			

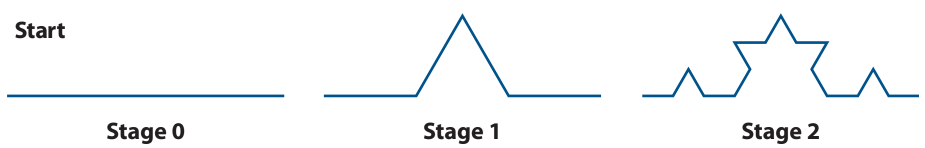
c.

x	0	1	2	3	4	5	6	7	8
y				48	56	64			

d.

x	0	1	2	3	4	5	6	7	8
y				125	625	3,125			

18 The diagram below shows the first stages in the formation of a geometric figure called a Koch curve. This figure is an example of a *fractal*. At each stage in the growth of the figure, the middle third of every segment is replaced by a "tent" formed by two equal-length segments. The new figure is made up of more, but shorter, segments of equal length.



a. Make a sketch showing at least one more stage in the growth of this fractal. Describe any symmetries that the fractal has at *each* stage.

b. Continue the pattern begun in this table:

Stage of Growth	0	1	2	3	4	5	6	7
Segments in Design	1	4						

c. Write a rule showing how the number of segments at any stage of the fractal can be used to find the number of segments at the next stage.

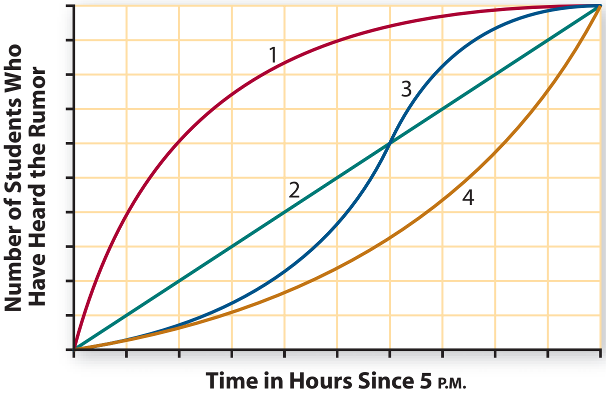
d. Write a rule that can be used to find the number of segments in the pattern at any stage x, without finding the numbers at each stage along the way. Begin your rule, "$y = \ldots$."

e. Use the rule from Part d to produce a table and a graph showing the number of segments in the fractal pattern at each of the first 15 stages of growth. At what stage will the number of segments in the fractal first reach or pass 1 million?

19 News stories spread rapidly in modern society. With broadcasts over television, radio, and the Internet, millions of people hear about important events within hours. The major news providers try hard to report only stories that they know are true. But quite often rumors get started and spread around a community by word of mouth alone.

Suppose that to study the spread of information through rumors, two students started this rumor at 5 P.M. one evening: "Because of the threat of a huge snowstorm, there will be no school tomorrow and probably for the rest of the week." The next day they surveyed students at the school to find out how many heard the rumor and when they heard it.

a. What pattern of rumor spread is suggested by each of the graphs below?



b. Which pattern of change in number of students who have heard the rumor is most likely to match experimental results in case:

i. the rumor is spread by word of mouth from one student to another?

ii. the rumor is mentioned on radio and television broadcasts between 5 and 6 P.M.?

Jeff Greenberg/Photo Edit

20 On one rainy day, the Silver Spring Soccer Club president decides that she will call all 750 member families herself to tell them about cancellation of play. She figures that she can make 3 calls per minute.

a. How long will it take her to notify all families in the club?

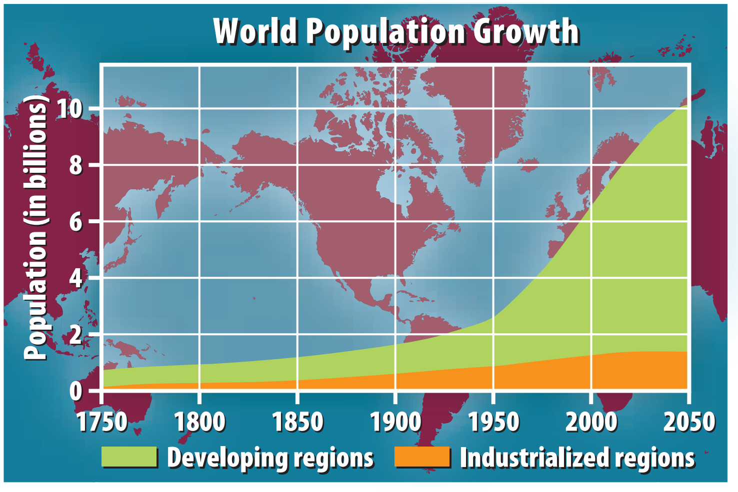
b. Look back at your results from work on Applications Task 2 and estimate the time it would take to inform all families of the cancellation if that calling tree was used instead.

21 Exponential functions, like linear functions, can be expressed by rules relating x and y values and by rules relating *NOW* and *NEXT* y values when x increases in steps of 1. Compare the patterns of (x, y) values produced by these functions: $y = 2(3^x)$ and $y = 2 + 3x$ by completing these tasks.

a. For each function, write another rule using *NOW* and *NEXT* that could be used to produce the same pattern of (x, y) values.

b. How would you describe the similarities and differences in the relationships of x and y in terms of their function graphs, tables, and rules?

22 The population of our world was about 6.5 billion in 2005. At the present rate of growth, that population will double approximately every 60 years. (Source: *The World Factbook 2005*. CIA.)



Sources: United Nations Population Division and Population Reference Bureau, 1993.

a. Assuming this rate continues, what will be the population 60, 120, 180, and 240 years from now?

b. How would that growth pattern compare to a pattern that simply added 6.5 billion people every 60 years?

c. Do you think the population is likely to continue growing in the "doubling every 60 years" pattern? Explain your reasoning.

d. How might rapid population growth affect your life in the next 60 years?

23 One way to think about rates of growth is to calculate the time it will take for a quantity to double in value. For example, it is common to ask how long it will take a bank investment or a country's population to double.

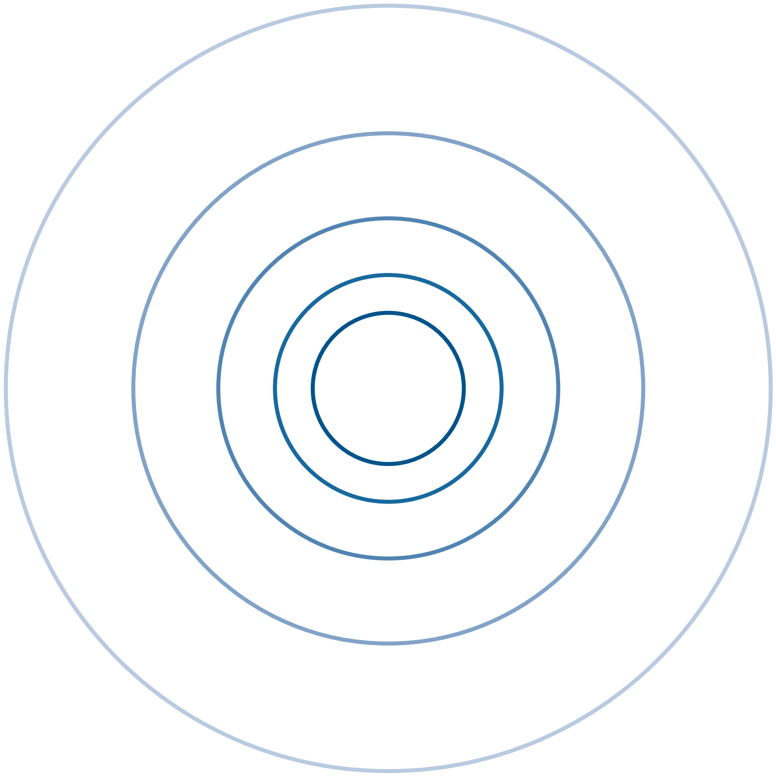
a. If the U.S. population in 2000 was about 276 million and growing exponentially at a rate of 0.9% per year, how long will it take for the U.S. population to double?

b. One year's growth is 0.9% of 276 million, or about 2.5 million. How long would it take the U.S. population to double if it increased *linearly* at the rate of 2.5 million per year?

c. How long does it take a bank deposit of $5,000 to double if it earns interest compounded annually at a rate of 2%? At a 4% rate? At a 6% rate? At an 8% rate? At a 12% rate?

d. Examine your (*rate, time to double*) data in Part c. What pattern suggests a way to predict the doubling time for an investment of $5,000 at an interest rate of 3% compounded annually? Check your conjecture. If your prediction was not close, search for another pattern for predicting doubling time and check it.

24 The sketch below shows a small circle of radius 10 millimeters and the results of four different size transformations—each with scale factor 1.5 applied to the previous circle.



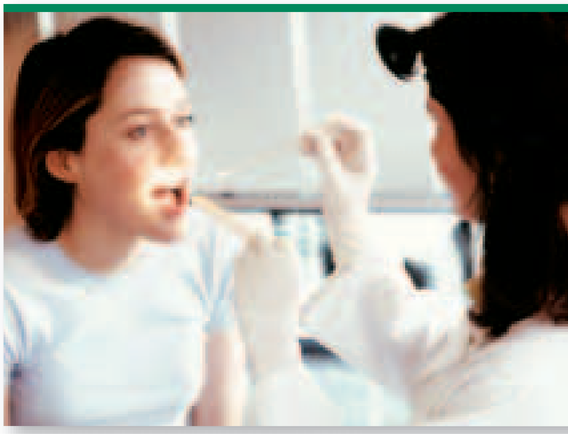
a. Make a table of (*radius, circumference*) values for the 5 circles and plot a graph of the resulting ordered pairs.

b. Write a *NOW-NEXT* rule showing how the circumference of each circle is related to the next larger circle.

c. If the pattern of expanding circles with the same center were continued, what rule would show how to calculate the circumference of the *n*th circle?

d. Enter the areas of the first 5 circles in the pattern in a new row of your table for Part a. Then write a rule that would show how to calculate the area enclosed by the *n*th circle in that pattern.

Reflections



25 One common illness in young people is *strep throat*. This bacterial infection can cause painful sore throats. Have you or anyone you know ever had strep throat? How does what you have learned about exponential growth explain the way strep throat seems to develop very quickly?

26 Which of the rules for exponential growth by doubling do you prefer: *NEXT* = 2 • *NOW* or $y = 2^x$? Give reasons for your preference and explain how the two types of rules are related to each other.

27 Exponential functions, like linear functions, can be expressed by rules relating x and y values.

a. In exponential functions with rules $y = a(b^x)$:

- **i.** how does the value of a affect the graph?
- **ii.** how does the value of b affect the graph?

b. In linear functions with rules $y = a + bx$:

- **i.** how does the value of a affect the graph?
- **ii.** how does the value of b affect the graph?

28 In each of the calculations in Parts a–d, a student has made an error. Correct the errors. Then describe how you might remind the student how to correctly calculate the answer.

a. $2^3 = 6$

b. $-2^2 = 4$

c. $3^0 = 0$

d. $(-2)^4 = -16$

29 Make a table to compare the values of 2^b and b^2 for several different positive integer values of b.

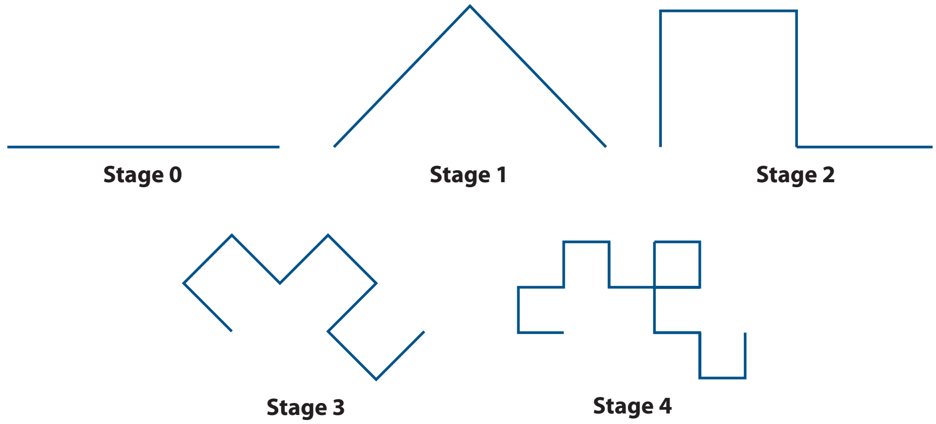
b								
2^b								
b^2								

a. For what values of b is $2^b = b^2$?

b. For what values of b is $2^b > b^2$? Explain the reasoning that supports your answer.

Extensions

30 The drawings below show five stages in growth of a design called the *dragon fractal*.



a. Draw Stage 5 of growth in the dragon fractal.

b. What pattern of change do you see in the number of segments of the growing fractal?

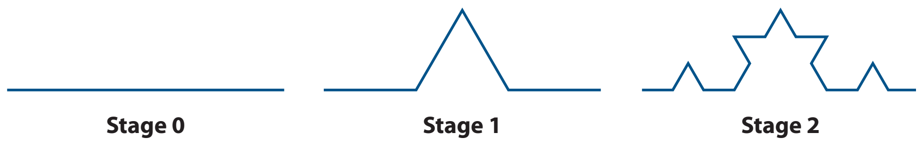
c. Make a table and a plot of the data showing that pattern of change.

d. Write a rule relating *NOW* and *NEXT* and a rule beginning "$y = \ldots$" for finding the number of segments in the figure at each stage of growth.

e. How many segments will there be in the fractal design at Stage 16?

f. At what stage will the fractal design have more than 1,000 segments of equal length?

31 In this task, you will examine more closely the Koch curve fractal from Connections Task 18.



Recall that in moving from one stage to the next, each segment is divided into three equal-length parts. A tent is raised over the middle section with sides equal in length to the parts on each side.

a. If the original line segment is 1 inch long, how long is each segment of the pattern in Stage 1? How long is each segment of the pattern in Stage 2?

b. Complete the following table showing the length of segments in the first 10 stages.

Stage	0	1	2	3	4	5	6	7	8	9
Length	1	$\frac{1}{3}$	$\frac{1}{9}$							

c. Look back at Parts c and d of Connections Task 18 where you wrote rules giving the number of short segments at each stage of the pattern. Then use that information and the results of Part b to complete the following table giving the total length of the pattern at each stage.

Stage	0	1	2	3	4	5	6	7	8	9
Length	1	$\frac{4}{3}$	$\frac{16}{9}$							

d. What appears to be happening to the total length of the pattern as the number of segments in the pattern increases?

32 Banks frequently pay interest more often than once each year. Suppose your bank pays interest compounded *quarterly*. If the annual percentage rate is 4%, then the bank adds 4% ÷ 4 = 1% interest to the account balance at the end of each 3-month period.

a. Explore the growth of a $1,000 deposit in such a bank over 5 years.

b. Compare the quarterly compounding with annual compounding at 4%.

c. Repeat the calculations and comparisons if the annual rate is 8%.

33 Many people borrow money from a bank to buy a car, a home, or to pay for a college education. However, they have to pay back the amount borrowed plus interest. To consider a simple case, suppose that for a car loan of $9,000 a bank charges 6% annual rate of interest compounded quarterly and the repayment is done in quarterly installments. One way to figure the balance on this loan at any time is to use the rule:

$$\textit{new balance} = 1.015 \times \textit{old balance} - \textit{payment}.$$

a. Use this rule to find the balance due on this loan for each quarterly period from 0 to 20, assuming that the quarterly payments are all $250.

b. Experiment with different payment amounts to see what quarterly payment will repay the entire $9,000 loan in 20 payments (5 years).

c. Create a spreadsheet you can use in experiments to find the effects of different quarterly payment amounts, interest rates, and car loan amounts. Use the spreadsheet to look for patterns relating those variables for 5-year loans. Write a brief report of your findings. (You might want to have cells in the spreadsheet for each of the variables, *original loan amount, interest rate*, and *quarterly payment amount*, and then a column that tracks the *outstanding loan balance* over a period of 20 quarters.)

34 The Wheaton Boys and Girls Club has 511 members and a calling tree in which, starting with the president, members are asked to pass on news to two other members.

a. What function rule shows how to calculate the total number T of members informed after x stages of the process have been completed? It might help to begin by finding these sums:

1

$1 + 2$

$1 + 2 + 4$

$1 + 2 + 4 + 8$

$1 + 2 + 4 + 8 + 16$

$\vdots$

Use the function rule you came up with to make a table and a graph showing values of T for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

b. What function rule gives the number of people N in the club who have not yet been called after stage x of the calling tree process? Use the function rule you came up with to make a table and graph showing values of N for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

c. Describe the patterns of change shown in tables and graphs of the rules that express T and N as functions of x and compare them to each other and to other functions you've encountered.

Review

35 Write each of the following calculations in more compact form by using exponents.

a. $5 \times 5 \times 5 \times 5$

b. $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

c. $1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5$

d. $(-10) \times (-10) \times (-10) \times (-10) \times (-10) \times (-10) \times (-10) \times (-10)$

e. $\underbrace{6 \times 6 \times \cdots \times 6}_{n \text{ factors}}$

f. $\underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}$

36 Do these calculations *without* use of the exponent key (^ or y^x) on your calculator.

a. 5^4 **b.** $(-7)^2$ **c.** 10^0

d. $(-8)^3$ **e.** 2^8 **f.** 2^{10}

On Your Own

37 Draw and label each triangle as accurately as you can. After you draw the triangle, use a ruler to determine the lengths of the remaining sides and a protractor to find the measures of the remaining angles.

	AB	*BC*	*CA*	m∠*ABC*	m∠*BCA*	m∠*CAB*
a.	2.5 in.	4 in.		125°		
b.	12 cm			45°		45°

38 Given that $\frac{1}{4}$ of 160 is 40, and that 10% of 160 is 16, find the values of the following without using a calculator. Explain how you obtained your answers.

a. $\frac{3}{4}$ of 160, $\frac{5}{4}$ of 160, and $1\frac{3}{4}$ of 160

b. 5% of 160, 95% of 160, and 105% of 160

39 Anthony took an inventory of the colors of the shirts that he owns and made a table of his findings.

Shirt Color	Green	Blue	Black	Red	Other
Number	5	2	4	3	6

a. Identify two types of graphs that would be appropriate for Anthony to make to display this data. Choose one of them and make it.

b. What percent of Anthony's shirts are red?

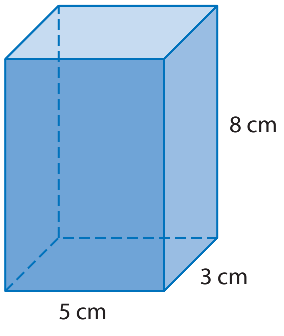
c. What percent of Anthony's shirts are not green?

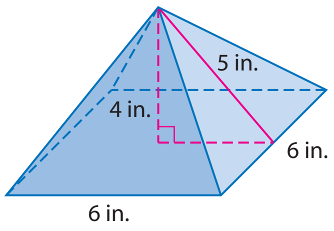
d. If Anthony were to randomly choose a shirt to wear, what is the probability that he would choose a black shirt?

40 A line passes through the points (1, 1) and (5, −7). Determine whether or not each point below is also on the line. Explain your reasoning.

a. (8, −12) **b.** (0, 3) **c.** (3, −4) **d.** (−3, 9)

41 Consider the two solids shown below.





For each solid, complete the following.

a. How many faces does the solid have? For each face, describe the shape and give its dimensions.

b. Identify all faces that appear to be parallel to each other.

c. Identify all faces that appear to be perpendicular to each other.

d. Find the surface area.

e. Find the volume. (Recall that the formula for the volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid.)

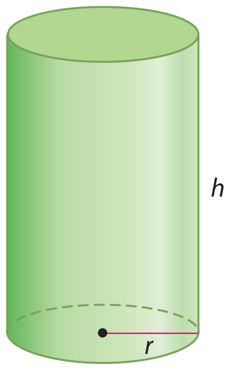
42 Evaluate each of these algebraic expressions when $x = 3$ and be prepared to explain why you believe you've produced the correct values.

a. $3x + 7$ **b.** $7 + 3x$ **c.** $5 - 4x + 3x^2$

d. $8(4x - 9)^2$ **e.** $\frac{5x - 2}{5 + 2x}$ **f.** $-2x^2$

g. $(-2x)^3$ **h.** $\sqrt{25 - x^2}$ **i.** $\frac{x + 12}{x + 4}$

43 When you use formulas to find values of dependent variables associated with specific values of independent variables, you need to be careful to "read" the directions of the formula the way they are intended. For example, the formula for surface area of a circular cylinder is $A = 2\pi r^2 + 2\pi rh$.



a. What is the surface area of a cylinder with radius 5 inches and height 8 inches? How do you know your calculation used the formula correctly?

b. Sketch a cylinder for which the height is the same length as the radius r. Show that the surface area of the cylinder is $A = 4\pi r^2$. What algebraic properties did you use in your reasoning?

44 Graph each of the following lines on grid paper. Then, for Parts a and b, write an equation of the line.

a. A line with slope of $\frac{2}{3}$ and y-intercept at (0, 6)

b. A line with slope of 0.5 and x-intercept at (−2, 0)

c. A line with equation $y = -3x - 5$

LESSON 2

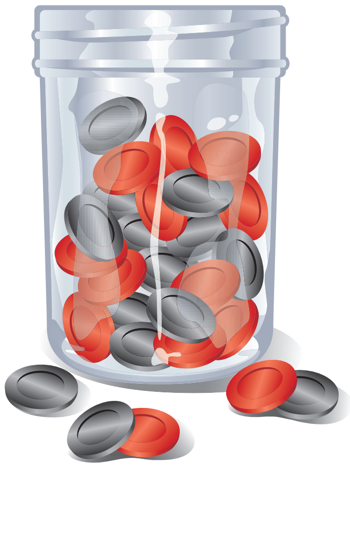
Exponential Decay

In 1989, the oil tanker Exxon Valdez ran aground in waters near the Kenai peninsula of Alaska. Over 10 million gallons of oil spread on the waters and shoreline of the area, endangering wildlife. That oil spill was eventually cleaned up—some of the oil evaporated, some was picked up by specially equipped boats, and some sank to the ocean floor as sludge. But the experience had lasting impact on thinking about environmental protection.

For scientists planning environmental cleanups, it is important to be able to predict the pattern of dispersion in such contaminating spills. Suppose that an accident dropped some pollutant into a large aquarium. It's not practical to remove all water from the aquarium at once, so the cleanup has to take place in smaller steps. A batch of polluted water is removed and replaced by clean water. Then the process is repeated.

Think about the following experiment that simulates pollution and cleanup of the aquarium.

- Mix 20 black checkers (the pollution) with 80 red checkers (the clean water).
- Remove 20 checkers from the mixture (without looking at the colors) and replace them with 20 red checkers (clean water). Record the number of black checkers remaining. Then shake the new mixture. This simulates draining off some of the polluted water and replacing it with clean water.

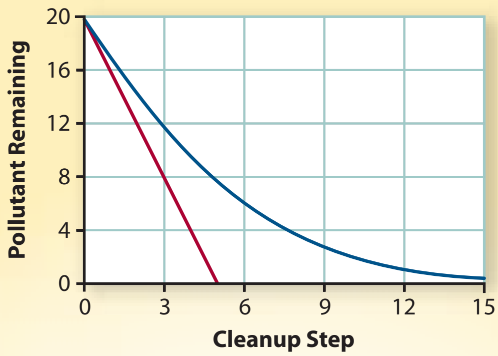


AP/Wide World Photos

- In the second step, remove 20 checkers from the new mixture (without looking at the colors) and replace them with 20 red checkers (more clean water). Record the number of black checkers remaining. Then stir the new mixture.
- Repeat the remove-replace-record-mix process for several more steps.

Think About This Situation

The graphs below show two possible outcomes of the pollution and cleanup simulation.



a What pattern of change is shown by each graph?

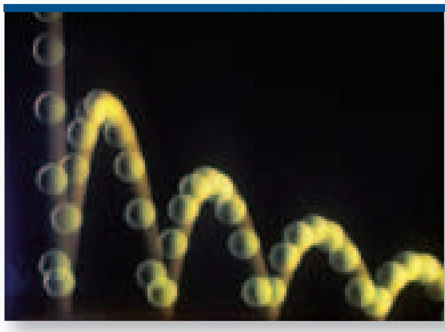
b Which graph shows the pattern of change that you would expect for this situation? Test your idea by running the experiment and plotting the (*cleanup step*, *pollutant remaining*) data.

c What sort of function relating pollution P and cleanup steps x would you expect to match your data plot? Test your idea using a graphing calculator or computer software.

The pollution cleanup experiment gives data in a pattern that occurs in many familiar and important problem situations. That pattern is called *exponential decay*. Your work on problems of this lesson will reveal important properties and uses of exponential decay functions and fractional exponents.

Investigation 1 More Bounce to the Ounce

Most popular American sports involve balls of some sort. In designing those balls, one of the most important factors is the bounciness or *elasticity* of the ball. For example, if a new golf ball is dropped onto a hard surface, it should rebound to about $\frac{2}{3}$ of its drop height. The pattern of change in successive rebound heights will be similar to that of the data in the pollution cleanup experiment.



Henry Groskinsky/Peter Arnold

As you work on the problems of this investigation, look for answers to this question:

What mathematical patterns in tables, graphs, and symbolic rules are typical of exponential decay relations?

1 Suppose a new golf ball drops downward from a height of 27 feet onto a paved parking lot and keeps bouncing up and down, again and again. Rebound height of the ball should be $\frac{2}{3}$ of its drop height. Make a table and plot of the data showing expected heights of the first ten bounces of the golf ball.

Bounce Number	0	1	2	3	4	5	6	7	8	9	10
Rebound Height (in feet)	27										

a. How does the rebound height change from one bounce to the next? How is that pattern shown by the shape of the data plot?

b. What rule relating *NOW* and *NEXT* shows how to calculate the rebound height for any bounce from the height of the preceding bounce?

c. What rule beginning "$y = \ldots$" shows how to calculate the rebound height after any number of bounces?

d. How will the data table, plot, and rules for calculating rebound height change if the ball drops first from only 15 feet?

As is the case with all mathematical models, data from actual tests of golf ball bouncing will not match exactly the predictions from rules about ideal bounces. You can simulate the kind of quality control testing that factories do by running some experiments in your classroom. Work with a group of three or four classmates to complete the next problems.

2 Get a golf ball and a tape measure or meter stick for your group. Decide on a method for measuring the height of successive rebounds after the ball is dropped from a height of at least 8 feet. Collect data on the rebound height for successive bounces of the ball.

a. Compare the pattern of your data to that of the model that predicts rebounds which are $\frac{2}{3}$ of the drop height. Would a rebound height factor other than $\frac{2}{3}$ give a better model for your data? Be prepared to explain your reasoning.

b. Write a rule using *NOW* and *NEXT* that relates the rebound height of any bounce of your tested ball to the height of the preceding bounce.

c. Write a rule beginning "$y = \ldots$" to predict the rebound height after any bounce.

Tim Fuller

3 Repeat the experiment of Problem 2 with some other ball such as a tennis ball or a volleyball.

a. Study the data to find a reasonable estimate of the rebound height factor for your ball.

b. Write a rule using *NOW* and *NEXT* and a rule beginning "$y = \ldots$" to model the rebound height of your ball on successive bounces.

Summarize the Mathematics

Different groups might have used different balls and dropped the balls from different initial heights. However, the patterns of (*bounce number*, *rebound height*) data should have some similar features.

a Look back at the data from your experiments.

- **i.** How do the rebound heights change from one bounce to the next in each case?
- **ii.** How is the pattern of change in rebound height shown by the shape of the data plots in each case?

b List the *NOW-NEXT* and the "$y = \ldots$" rules you found for predicting the rebound heights of each ball on successive bounces.

- **i.** What do the rules relating *NOW* and *NEXT* bounce heights have in common in each case? How, if at all, are those rules different, and what might be causing the differences?
- **ii.** What do the rules beginning "$y = \ldots$" have in common in each case? How, if at all, are those rules different, and what might be causing the differences?

c What do the tables, graphs, and rules in these examples have in common with those of the exponential growth examples in Lesson 1? How, if at all, are they different?

d How are the exponential decay data patterns, graphs, and rules similar to and different from those of linear functions and other types of functions you've studied in earlier units?

Be prepared to compare your data, models, and ideas with the rest of the class.

Check Your Understanding

When dropped onto a hard surface, a brand new softball should rebound to about $\frac{2}{5}$ the height from which it is dropped.

a. If the softball is dropped 25 feet from a window onto concrete, what pattern of rebound heights can be expected?

- **i.** Make a table and plot of predicted rebound data for 5 bounces.
- **ii.** What *NOW-NEXT* and "$y = \ldots$" rules give ways of predicting rebound height after any bounce?

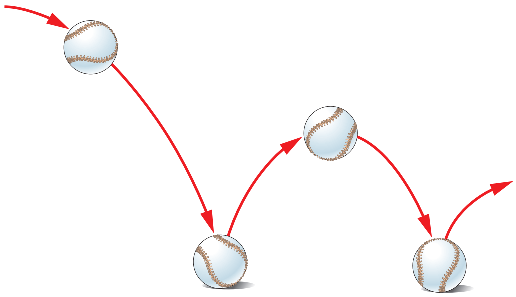
b. Here are some data from bounce tests of a softball dropped from a height of 10 feet.

Bounce Number	1	2	3	4	5
Rebound Height (in feet)	3.8	1.3	0.6	0.2	0.05

- **i.** What do these data tell you about the quality of the tested softball?
- **ii.** What bounce heights would you expect from this ball if it were dropped from 20 feet instead of 10 feet?

c. What *NOW-NEXT* and "$y = \ldots$" rules would model rebound height of an ideal softball if the drop were from 20 feet?

d. What rule beginning "$y = \ldots$" shows how to calculate the height y of the rebound when a new softball is dropped from any height x? What connections do you see between this rule and the rule predicting rebound height on successive bounces of the ball?



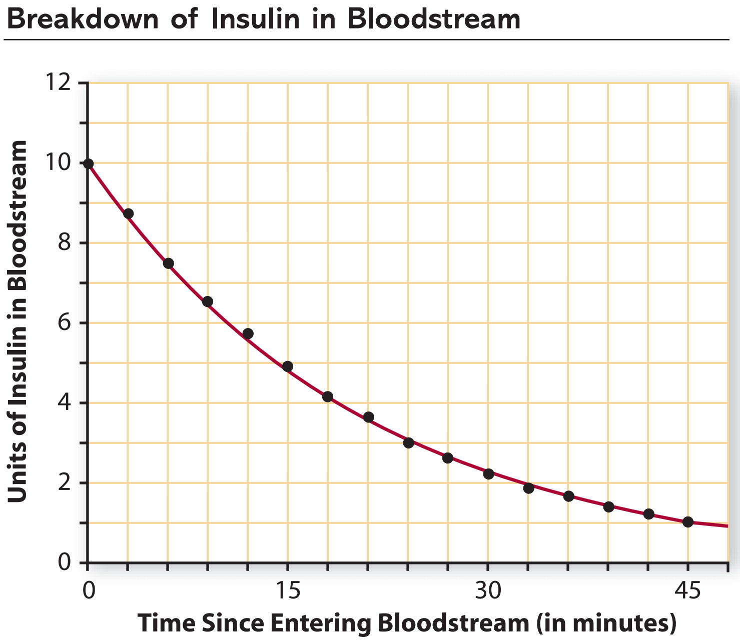
Investigation 2 Medicine and Mathematics

Prescription drugs are a very important part of the human health equation. Many medications are essential in preventing and curing serious physical and mental illnesses.

Diabetes, a disorder in which the body cannot metabolize glucose properly, affects people of all ages. In 2005, there were about 14.6 million diagnosed cases of diabetes in the United States. It was estimated that another 6.2 million cases remained undiagnosed. (Source: diabetes.niddk.nih.gov/dm/pubs/statistics/index.htm)

In 5–10% of the diagnosed cases, the diabetic's body is unable to produce insulin, which is needed to process glucose.

To provide this essential hormone, these diabetics must take injections of a medicine containing insulin. The medications used (called insulin delivery systems) are designed to release insulin slowly. The insulin itself breaks down rather quickly. The rate varies greatly among individuals, but the following graph shows a typical pattern of insulin decrease.



As you work on the problems of this investigation, look for answers to the following questions:

How can you interpret and estimate or calculate values of expressions involving fractional or decimal exponents?

How can you interpret and estimate or calculate the half-life of a substance that decays exponentially?

1. Medical scientists often are interested in the time it takes for a drug to be reduced to one half of the original dose. They call this time the **half-life** of the drug. What appears to be the half-life of insulin in this case?

2. The pattern of decay shown on this graph for insulin can be modeled well by the function $y = 10(0.95^x)$, where x is the number of minutes since the insulin entered the bloodstream.

   a. Use your calculator or computer software to see how well a table of values and graph of this rule matches the pattern in the graph above.

   b. What do the numbers 10 and 0.95 tell about the amount of insulin in the bloodstream?

   c. Based on the function modeling insulin decay, what percent of active insulin is actually used up with each passing minute?

3. What rule relating *NOW* and *NEXT* shows how the amount of insulin in the blood changes from one minute to the next, once 10 units have entered the bloodstream?

4. The insulin decay graph shows data points for three-minute intervals following the original insulin level. But the curve connecting those points reminds us that the insulin breakdown does not occur in sudden bursts at the end of each minute! It occurs *continuously* as time passes.

What would each of the following calculations tell about the insulin decay situation? Based on the graph, what would you expect as reasonable values for those calculations?

a. $10(0.95)^{1.5}$ **b.** $10(0.95)^{4.5}$ **c.** $10(0.95)^{18.75}$

5 Mathematicians have figured out ways to do calculations with fractional or decimal exponents so that the results fit into the pattern for whole number exponents. One of those methods is built into your graphing calculator or computer software.

a. Enter the function $y = 10(0.95^x)$ in your calculator or computer software. Then complete a copy of the following table of values showing the insulin decay pattern at times other than whole-minute intervals.

Elapsed Time (in minutes)	0	1.5	4.5	7.5	10.5	13.5	16.5	19.5
Units of Insulin in Blood	10							

b. Compare the entries in this table with data shown by points on the graph on the preceding page.

c. Study tables and graphs of your function to estimate, to the nearest tenth of a minute, solutions for the following equations and inequality. In each case, be prepared to explain what the solution tells about decay of insulin.

i. $2 = 10(0.95^x)$ **ii.** $8 = 10(0.95^x)$

iii. $10(0.95^x) > 1.6$

6 Use the function $y = 10(0.95^x)$ to estimate the half-life of insulin for an initial dose of 10 units. Then estimate the half-life in cases when the initial dose is 15 units. When it is 20 units. When it is 25 units. Explain the pattern in those results.

Summarize the Mathematics

In this investigation, you have seen another example of the way that patterns of exponential decay can be expressed by function rules like $y = a(b^x)$.

a What *NOW-NEXT* rule describes this pattern of change?

b What do the values of a and b tell about the situation being modeled? About the tables and graphs of the (x, y) values?

c How can you estimate or calculate values of b^x when x is not a whole number?

d What does the half-life tell about a substance that decays exponentially? What strategies can be used to estimate or calculate half-life?

Be prepared to compare your responses with those of your classmates.

Check Your Understanding

The most famous antibiotic drug is penicillin. After its discovery in 1929, it became known as the first *miracle drug*, because it was so effective in fighting serious bacterial infections.

Drugs act somewhat differently on each person. But, on average, a dose of penicillin will be broken down in the blood so that one hour after injection only 60% will remain active. Suppose a patient is given an injection of 300 milligrams of penicillin at noon.

a. Write a rule in the form $y = a(b^x)$ that can be used to calculate the amount of penicillin remaining after any number of hours x.

b. Use your rule to graph the amount of penicillin in the blood from 0 to 10 hours. Explain what the pattern of that graph shows about the rate at which active penicillin decays in the blood.

c. Use the rule from Part a to produce a table showing the amount of active penicillin that will remain at *quarter-hour* intervals from noon to 5 P.M.

- **i.** Estimate the half-life of penicillin.
- **ii.** Estimate the time it takes for an initial 300-mg dose to decay so that only 10 mg remain active.

d. If 60% of a penicillin dose remains active one hour after an injection, what percent has been broken down in the blood?

Investigation 3 Modeling Decay

When you study a situation in which data suggest a dependent variable decreasing in value as a related independent variable increases, there are two strategies for finding a good algebraic model of the relationship. In some cases, it is possible to use the problem conditions and reasoning to determine the type of function that will match dependent to independent variable values. In other cases, some trial-and-error exploration or use of calculator or computer curve-fitting software will be necessary before an appropriate model is apparent.

From a scientific point of view, it is always preferable to have some logical explanation for choice of a model. Then the experimental work is supported by understanding of the relationship being studied. As you work on the following problems, look for answers to these questions:

What clues in problem conditions are helpful in deriving function models for experimental data involving decay?

How can logical analysis of an experiment be used as a check of a function model produced by your calculator or computer curve-fitting software?

1. Suppose that you were asked to conduct this experiment:
   - Get a collection of 100 coins, shake them well, and drop them on a tabletop.
   - Remove all coins that are lying heads up and record the number of coins left.
   - Repeat the shake-drop-remove-record process until 5 or fewer coins remain.

   a. If you were to record the results of this experiment in a table of (*drop number, coins left*) values, what pattern would you expect in the data? What function rule would probably be the best model relating drop number n to number of coins left c?

   b. Conduct the experiment, record the data, and then use your calculator or curve-fitting software to find a function model that seems to fit the data pattern well.

   c. Compare the model suggested by logical analysis of the experiment to that found by fitting a function to actual data. Decide which you think is the better model of the experiment and be prepared to explain your choice.

2. Suppose that the experiment in Problem 1 is modified in this way:
   - Get a collection of 100 coins and place them on a table top.
   - Roll a six-sided die and remove the number of coins equal to the number on the top face of the die. Record the number of coins remaining. For example, if the first roll shows 4 dots on the top of the die, remove four coins, leaving 96 coins still on the table.
   - Repeat the roll-remove-record process until 10 or fewer coins remain.

   a. If you were to record the results of this experiment in a table of (*roll number, coins left*) values, what pattern would you expect in that data? What function rule would probably be the best model relating roll number n to number of coins left c?

   b. Conduct the experiment, record the data, and then use your calculator or curve-fitting software to find a function model that seems to fit the data pattern well.

   c. Compare the model suggested by logical analysis of the experiment to that found by fitting a function to actual data. Decide which you think is the better model of the experiment and be prepared to explain your choice.

3. How are the data from the experiments in Problems 1 and 2 and the best-fitting function models for those data different? Why are those differences reasonable, in light of differences in the nature of the experiments that were conducted?

Summarize the Mathematics

In this investigation, you compared two strategies for developing models of patterns in experimental data where a dependent variable decreases in value as a related independent variable increases.

a What differences did you notice between models suggested by logical analysis of the experiments and by curve-fitting based on real data?

b How were the models for each experiment similar, and how were they different? How are those similarities and differences explained by logical analysis? How are they illustrated by patterns in experimental data plots?

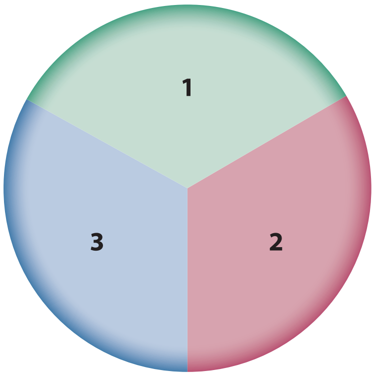
c What kinds of problem conditions suggest situations in which a linear model is likely to be best? Situations in which an exponential model is likely to be best?

Be prepared to compare your responses with those from other groups.

Check Your Understanding

Consider the following experiment:

- Start with a pile of 90 kernels of unpopped popcorn or dry beans.
- Pour the kernels or beans onto the center of a large paper plate with equal-sized sectors marked as in the diagram below.



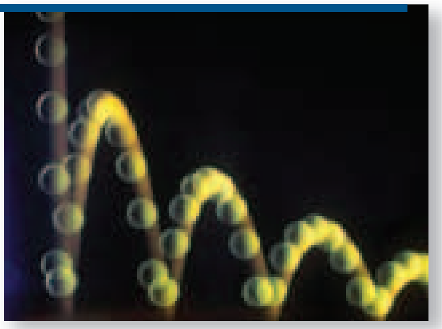
- Shake the plate so that the kernels or beans scatter into the various sectors in a somewhat random pattern.
- Remove all kernels that land on the sector marked "1" and record the trial number and the number of kernels or beans remaining.
- Repeat the shake-remove-record process several times.

a. If you were to record the results of this experiment in a table of (*trial number, kernels left*) values, what pattern would you expect in that data? What function rule would probably be the best model for the relationship between trial number n and kernels left k?

b. Conduct the experiment and record the data. Then use your calculator or curve-fitting software to find the model that seems to fit the data pattern well.

c. Compare the models suggested by logical analysis of the experiment and by fitting of a function to actual data. Decide which is the better model of the experiment and explain your choice.

Investigation 4 Properties of Exponents II



In studying the rebound height of a bouncing ball, you calculated powers of the fraction $\frac{2}{3}$. You can calculate a power like $\left(\frac{2}{3}\right)^4$ by repeated multiplication $\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$. But there is a shortcut rule for such calculations with exponents.

As you work on the problems in this investigation, make notes of answers to this question:

What exponent properties provide shortcut rules for calculating powers of fractions, quotients of powers, and negative exponents?

Powers of a Fraction As you work on the next calculations, look for a pattern suggesting ways to write powers of fractions in useful equivalent forms.

1 Find values of x and y that will make these equations true statements:

a. $\left(\frac{3}{5}\right)^3 = \frac{3^x}{5^y}$

b. $\left(\frac{c}{5}\right)^2 = \frac{c^x}{5^y}$

c. $\left(\frac{4}{n}\right)^5 = \frac{4^x}{n^y}$ $(n \neq 0)$

d. $\left(\frac{c^2}{n}\right)^3 = \frac{c^x}{n^y}$ $(n \neq 0)$

2 Examine the results of your work on Problem 1.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating powers of a fraction like $\left(\frac{3}{5}\right)^4$? Explain how you would help someone who made those errors correct their understanding of how exponents work.

Quotients of Powers Since many useful algebraic functions require division of quantities, it is helpful to be able to simplify expressions involving quotients of powers like $\frac{b^x}{b^y}$ $(b \neq 0)$.

3 Find values for x, y, and z that will make these equations true statements.

a. $\frac{2^{10}}{2^3} = 2^z$ **b.** $\frac{3^6}{3^2} = 3^z$ **c.** $\frac{10^9}{10^3} = 10^z$ **d.** $\frac{2^x}{2^5} = 2^7$

e. $\frac{7^x}{7^y} = 7^2$ **f.** $\frac{b^5}{b^3} = b^z$ **g.** $\frac{3^5}{3^5} = 3^z$ **h.** $\frac{b^x}{b^x} = b^z$

4 Examine the results of your work on Problem 3.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating quotients of powers like $\frac{8^{12}}{8^4}$? Explain how you would help someone who made those errors correct their understanding of how exponents work.

5 Use your answers to Problem 3 Parts g and h and Problem 4 Part a to explain why it is reasonable to define $b^0 = 1$ for any base b $(b \neq 0)$.

Negative Exponents Suppose that you were hired as a science lab assistant to monitor an ongoing experiment studying the growth of an insect population. If the population when you took over was 48 and it was expected to double every day, you could estimate the population for any time in the future or the past with the function $p = 48(2^x)$.

Future estimates are easy: One day from now, the population should be about $48(2^1) = 96$; two days from now it should be about $48(2^2) = 48(2)(2) = 192$, and so on.

Estimates of the insect numbers in the population before you took over require division: One day earlier, the population should have been about $48(2^{-1}) = 48 \div 2 = 48\left(\frac{1}{2}\right) = 24$; two days ago, it should have been about:

$$\begin{aligned} 48(2^{-2}) &= (48 \div 2) \div 2 \\ &= 48 \div 2^2 \\ &= 48\left(\frac{1}{2^2}\right) \\ &= 12 \end{aligned}$$

This kind of reasoning about exponential growth suggests a general rule that for any nonzero number b and any integer n, $b^{-n} = \frac{1}{b^n}$.

6 The rule for operating with negative integer exponents also follows logically from the property about quotients of powers and the definition $b^0 = 1$. Justify each step in the reasoning below.

$$\begin{aligned} \frac{1}{b^n} &= \frac{b^0}{b^n} && (1) \\ &= b^{0-n} && (2) \\ &= b^{-n} && (3) \end{aligned}$$

7 Use the relationship between fractions and negative integer exponents to write each of the following expressions in a different but equivalent form. In Parts a–f, write an equivalent fraction that does not use exponents at all.

a. 5^{-3} **b.** 6^{-1} **c.** 2^{-4} **d.** $\left(\frac{2}{5}\right)^{-1}$

e. $\left(\frac{1}{2}\right)^{-3}$ **f.** $\left(\frac{2}{5}\right)^{-2}$ **g.** x^{-3} **h.** $\frac{1}{a^4}$

8 Examine the results of your work in Problems 6 and 7.

a. How would you describe the rule defining negative integer exponents in your own words?

b. What would you expect to see as the most common errors in evaluating expressions with negative integer exponents like $\left(\frac{4}{3}\right)^{-2}$? How would you help someone who made those errors correct their understanding of how negative integer exponents work?

Summarize the Mathematics

In this investigation, you discovered, tested, and justified several principles that allow writing of exponential expressions in convenient equivalent forms.

a How would you describe in words the properties for writing exponential expressions in equivalent forms that you discovered in work on Problems 1–8?

b Summarize the properties of exponents you explored in Problems 1–8 by completing each of these statements with equivalent exponential expressions. In each case, $b \neq 0$.

i. $\left(\frac{a}{b}\right)^m = \ldots$ **ii.** $\frac{b^m}{b^n} = \ldots$ **iii.** $b^{-n} = \ldots$

c What examples would you use to illustrate common errors in use of exponents in expressions like those of Part b, and how would you explain the errors in each example?

Be prepared to explain your ideas to the entire class.

Check Your Understanding

Use properties of exponents to write each of the following expressions in another equivalent form and be prepared to explain how you know your answers are correct.

a. $(y^3)(y^6)$ **b.** $(5x^2y^4)(2xy^3)$ **c.** $\frac{a^7}{a^5}$

d. $\left(\frac{5}{3}\right)^3$ **e.** $(pq)^3$ **f.** $(7p^3q^2)^2$

g. $(T^3)^2$ **h.** $\frac{2}{p^{-4}}$ **i.** -5^2

j. $(-5)^2$ **k.** $2a^0$ **l.** $(2a)^0$

m. $4a^{-2}$ **n.** $(4a)^{-2}$

Investigation 5 Square Roots and Radicals

In your work on problems of insulin decay, you found that some questions required calculation with exponential expressions involving a fractional base and fractional powers. For example, estimating the amount of insulin active in the bloodstream 1.5 minutes after a 10-unit injection required calculating $10(0.95^{1.5})$.

Among the most useful expressions with fractional exponents are those with power one-half. It turns out that one-half powers are connected to the square roots that are so useful in geometric calculations like those involving the Pythagorean Theorem. For any non-negative number b,

$$b^{\frac{1}{2}} = \sqrt{b}.$$

Expressions like $\sqrt{b}$, $\sqrt{5}$, and $\sqrt{9 - x^2}$ are called *radicals*. As you work on the following problems, keep this question in mind:

How can you use your understanding of properties of exponents to guide your thinking about one-half powers, square roots, radical expressions, and rules for operating with them?

1 For integer exponents m and n, you know that $(a^m)^n = a^{mn}$. That property can be extended to work with fractional exponents.

a. Write each of these expressions in standard number form without exponents or radicals.

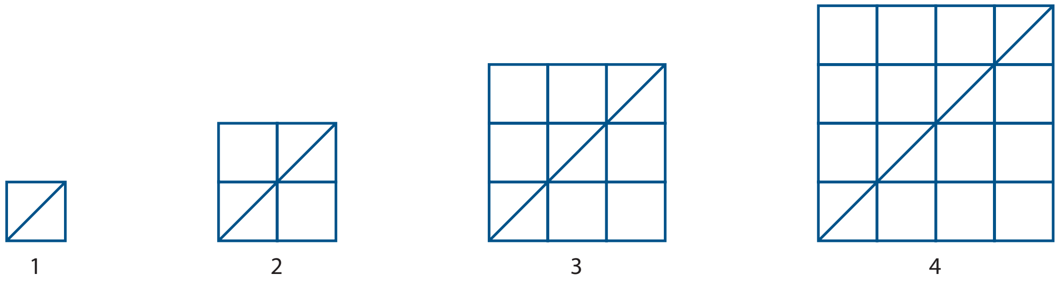
i. $\left(2^{\frac{1}{2}}\right)^2$ **ii.** $\left(5^{\frac{1}{2}}\right)^2$ **iii.** $\left(12^{\frac{1}{2}}\right)^2$ **iv.** $\left(2.4^{\frac{1}{2}}\right)^2$

b. How do the results of Part a explain why the definition $b^{\frac{1}{2}} = \sqrt{b}$ makes sense?

2 Write each of the following expressions in an equivalent form using radicals and then in simplest number form (without exponents or radicals).

a. $(25)^{\frac{1}{2}}$ **b.** $(9)^{\frac{1}{2}}$ **c.** $\left(\frac{9}{4}\right)^{\frac{1}{2}}$ **d.** $(100)^{\frac{1}{2}}$

3 The diagram below shows a series of squares with side lengths increasing in sequence 1, 2, 3, 4, and one diagonal drawn in each square.



a. Use the Pythagorean Theorem to find the exact length of the diagonal of each square.

b. How are the lengths of the diagonals in the three larger squares related to the length of the diagonal of the unit square?

c. Look for a pattern in the results of Part b to complete the statement beginning:

The length d of each diagonal in a square with sides of length s is given by d = … .

4 The pattern relating side and diagonal lengths in a square illustrates a useful rule for simplifying radical expressions:

For any non-negative numbers a and b: $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

a. What properties of square roots and exponents justify the steps in this argument? For any non-negative numbers a and b:

$$\begin{aligned} \sqrt{ab} &= (ab)^{\frac{1}{2}} && (1) \\ &= a^{\frac{1}{2}}b^{\frac{1}{2}} && (2) \\ &= \sqrt{a}\sqrt{b} && (3) \end{aligned}$$

b. Modify the argument in Part a to justify this property of radicals:

For any non-negative numbers a and b $(b \neq 0)$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

5 Use the properties of square roots in Problem 4 to write expressions a–h in several equivalent forms. In each case, try to find the simplest equivalent form—one that involves only one radical and the smallest possible number inside that radical. Check your ideas with calculator estimates of each form. For example,

$$\begin{aligned} \sqrt{48} &= \sqrt{4}\sqrt{12} \\ &= 2\sqrt{12} \\ &= 2\sqrt{4}\sqrt{3} \\ &= 2 \cdot 2\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

Calculator estimates show that $\sqrt{48} \approx 6.93$ and $4\sqrt{3} \approx 6.93$.

a. $\sqrt{9 \cdot 5}$ **b.** $\sqrt{18}\sqrt{8}$ **c.** $\sqrt{45}$ **d.** $\sqrt{4 \cdot 9}$

e. $\sqrt{4 \cdot \frac{1}{9}}$ **f.** $\sqrt{\frac{9}{4}}$ **g.** $\sqrt{12}$ **h.** $\sqrt{96}$

6 The properties of square roots in Problem 4 are like distributive properties—taking the square root distributes over the product or the quotient of two (or more) numbers. One of the most common errors in working with square roots is distributing the square root sign over addition. However,

$$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$$

except in some very special cases. Use several pairs of positive values for a and b to show that taking square roots *does not* distribute over addition (or subtraction).

Summarize the Mathematics

In this investigation, you explored the relationship between fractional exponents and square roots and important properties of radical expressions.

a For $n \geq 0$, what does $\sqrt{n}$ mean, and why does it make sense that $\sqrt{n} = n^{\frac{1}{2}}$?

b What property of square roots can be used to express $\sqrt{n}$ in equivalent, often simpler, forms?

c What formula gives the length of each diagonal in a square with sides of length s?

Be prepared to share your thinking with the entire class.

Check Your Understanding

Use your understanding of fractional exponents and radical expressions to help complete the following tasks.

a. How could you use a calculator with only +, −, ×, and ÷ keys to check these claims about values of expressions involving fractional exponents?

i. $225^{\frac{1}{2}} = 15$ **ii.** $7^{\frac{1}{2}} \approx 2.65$

b. Find the values of these expressions, without use of a calculator.

i. $36^{\frac{1}{2}}$ **ii.** $\sqrt{81}$

iii. $\left(\frac{25}{16}\right)^{\frac{1}{2}}$ **iv.** $\sqrt{\frac{49}{81}}$

c. Use the property that for non-negative numbers a and b, $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to help write each of these radical expressions in at least two equivalent forms.

i. $\sqrt{30}$ **ii.** $\sqrt{10}\sqrt{40}$

iii. $\sqrt{\frac{7}{25}}$ **iv.** $\sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}}$

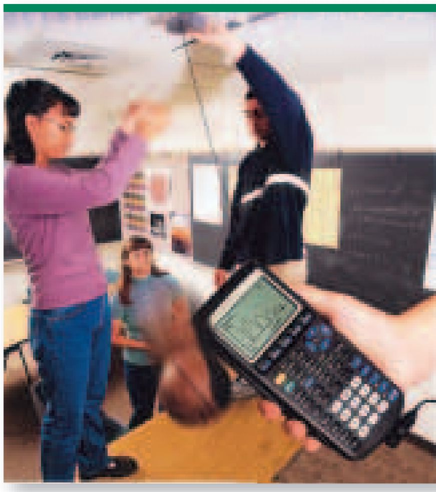
d. What is the length of each diagonal in a square with sides of length 7 centimeters?

e. Give a counterexample to show that for nonnegative numbers a and b, $\sqrt{a - b}$ is *not* equal to $\sqrt{a} - \sqrt{b}$

On Your Own

Applications

1 If a basketball is properly inflated, it should rebound to about $\frac{1}{2}$ the height from which it is dropped.



a. Make a table and plot showing the pattern to be expected in the first 5 bounces after a ball is dropped from a height of 10 feet.

b. At which bounce will the ball first rebound less than 1 foot? Show how the answer to this question can be found in the table and on the graph.

c. Write a rule using *NOW* and *NEXT* and a rule beginning "$y = \ldots$" that can be used to calculate the rebound height after many bounces.

d. How will the data table, plot, and rules change for predicting rebound height if the ball is dropped from a height of 20 feet?

e. How will the data table, plot, and rules change for predicting rebound height if the ball is somewhat over-inflated and rebounds to $\frac{3}{5}$ of the height from which it is dropped?

2 Records at the Universal Video store show that sales of new DVDs are greatest in the first month after the release date. In the second month, sales are usually only about one-third of sales in the first month. Sales in the third month are usually only about one-third of sales in the second month, and so on.

a. If Universal Video sells 180 copies of one particular DVD in the first month after its release, how many copies are likely to be sold in the second month? In the third month?

b. What *NOW-NEXT* and "$y = \ldots$" rules predict the sales in the following months?

c. How many sales are predicted in the 12th month?

d. In what month are sales likely to first be fewer than 5 copies?

e. How would your answers to Parts a–d change for a different DVD that has first-month sales of 450 copies?

3. You may have heard of athletes being disqualified from competitions because they have used anabolic steroid drugs to increase their weight and strength. These steroids can have very damaging side effects for the user. The danger is compounded by the fact that these drugs leave the human body slowly. With an injection of the steroid *cyprionate*, about 90% of the drug and its by-products will remain in the body one day later. Then 90% of that amount will remain after a second day, and so on. Suppose that an athlete tries steroids and injects a dose of 100 milligrams of cyprionate. Analyze the pattern of that drug in the athlete's body by completing the next tasks.

   a. Make a table showing the amount of the drug remaining at various times.

Time Since Use (in days)	0	1	2	3	4	5	6	7
Steroid Present (in mg)	100	90	81					

   b. Make a plot of the data in Part a and write a short description of the pattern shown in the table and the plot.

   c. Write two rules that describe the pattern of amount of steroid in the blood.

      i. Write a *NOW-NEXT* rule showing how the amount of steroid present changes from one day to the next.

      ii. Write a "$y = \ldots$" rule that shows how one could calculate the amount of steroid present after any number of days.

   d. Use one of the rules in Part c to estimate the amount of steroid left after 0.5 and 8.5 days.

   e. Estimate, to the nearest tenth of a day, the half-life of cyprionate.

   f. How long will it take the steroid to be reduced to only 1% of its original level in the body? That is, how many days will it take until 1 milligram of the original dose is left in the body?

4. When people suffer head injuries in accidents, emergency medical personnel sometimes administer a paralytic drug to keep the patient immobile. If the patient is found to need surgery, it's important that the immobilizing drug decay quickly.

   For one typical paralytic drug, the standard dose is 50 micrograms. One hour after the injection, half the original dose has decayed into other chemicals. The halving process continues the next hour, and so on.

   a. How much of the drug will remain in the patient's system after 1 hour? After 2 hours? After 3 hours?

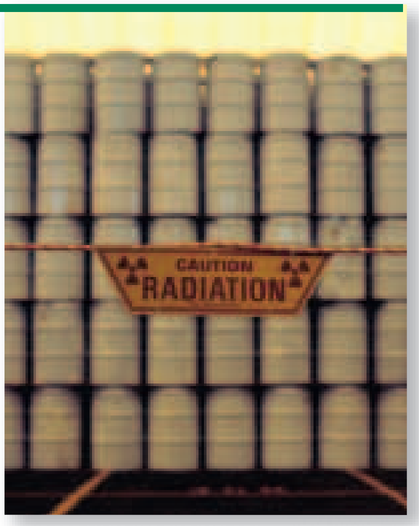
   b. Write a rule that shows how to calculate the amount of drug that will remain x hours after the initial dose.

   c. Use your rule to make a table showing the amount of drug left at half-hour intervals from 0 to 5 hours.

   d. Make a plot of the data from Part c and a continuous graph of the function on the same axes.

   e. How long will it take the 50-microgram dose to decay to less than 0.05 microgram?

Alamy Images



5 Radioactive materials have many important uses in the modern world, from fuel for power plants to medical x-rays and cancer treatments. But the radioactivity that produces energy and tools for "seeing" inside our bodies can have some dangerous effects too; for example, it can cause cancer in humans.

The radioactive chemical *strontium-90* is produced in many nuclear reactions. Extreme care must be taken in transport and disposal of this substance. It decays slowly—if an amount is stored at the beginning of a year, 98% of that amount will still be present at the end of the year.

a. If 100 grams (about 0.22 pound) of strontium-90 are released by accident, how much of that radioactive substance will still be around after 1 year? After 2 years? After 3 years?

b. Write two different rules that can be used to calculate the amount of strontium-90 remaining from an initial amount of 100 grams at any year in the future.

c. Make a table and a graph showing the amount of strontium-90 that will remain from an initial amount of 100 grams at the end of every 10-year period during a century.

Years Elapsed	0	10	20	30	40	50	...
Amount Left (in g)	100						

d. Find the amount of strontium-90 left from an initial amount of 100 grams after 15.5 years.

e. Find the number of years that must pass until only 10 grams remain.

f. Estimate, to the nearest tenth of a year, the half-life of strontium-90.

6 The values of expensive products like automobiles *depreciate* from year to year. One common method for calculating the depreciation of automobile values assumes that a car loses 20% of its value every year. For example, suppose a new pickup truck costs \$20,000. The value of that truck one year later will be only $20{,}000 - 0.2(20{,}000) = \$16{,}000$.

a. Why is it true that for any value of x, $x - 20\%x = 80\%x$? How does this fact provide two different ways of calculating depreciated values?

b. Write *NOW-NEXT* and "$y = \ldots$" rules that can be used to calculate the value of the truck in any year.

c. Estimate the time when the truck's value is only \$1,000. Show how the answer to this question can be found in a table and on a graph.

d. How would the rules in Part b change if the truck's purchase price was only \$15,000? What if the purchase price was \$25,000?

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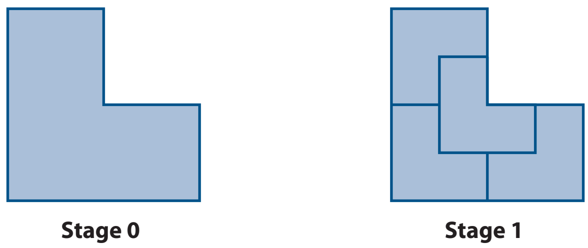
CPMP-Tools

7 In Applications Task 4 of Lesson 1, you counted the number of "chairs" at each stage in a design process that begin like this:



The chair at Stage 0 can be made by placing three square tiles in an "L" pattern. Suppose that the tiles used to make the chair design at Stage 0 are each one-centimeter squares. Then the left side and the bottom of that chair are each two centimeters long.

a. Complete a table like this that shows the lengths of those chair sides in smaller chairs used at later stages of the subdivision process.

Subdivision Stage	0	1	2	3	4	5	…	*n*
Side Length (in cm)	2	…						

b. Write two rules that show how to calculate the side length (in cm) of the smaller chair at any stage—one using *NOW* and *NEXT*, and another beginning "$L = \ldots$."

c. The area of the chair at Stage 0 is 3 square centimeters. What is the area of each small chair at Stage 1? At Stage 2? At Stage 3? At Stage n?

d. Write two rules that show how to calculate the area (in cm^2) of the smaller chairs at any stage—one using *NOW* and *NEXT*, and another beginning "$A = \ldots$."

8 Fleas are one of the most common pests for dogs. If your dog has fleas, you can buy many different kinds of treatments, but they wear off over time. Suppose the half-life of one such treatment is 10 days.

a. Make a table showing the fraction of an initial treatment that will be active after 10, 20, 30, and 40 days.

b. Experiment with your calculator or computer software to find a function of the form $y = b^x$ (where x is time in days) that matches the pattern in your table.

On Your Own

9. Suppose that an experiment to test the bounce of a tennis ball gave the data in the following table.

Bounce Number	1	2	3	4	5	6
Bounce Height (in inches)	35	20	14	9	5	3

a. Find *NOW-NEXT* and "$y = \ldots$" rules that model the relationship between bounce height and bounce number shown in the experimental data.

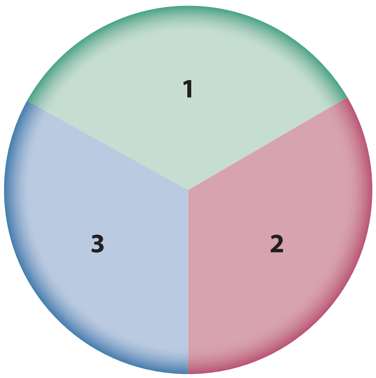
b. Use either rule from Part a to estimate the drop height of the ball.

c. Modify the rules from Part a to provide models for the relationship between bounce height and bounce number in case the drop height was 100 inches. Then make a table and plot of estimates for the heights of the first 6 bounces in this case.

d. What percent seems to describe well the relationship between drop height and bounce height of the tennis ball used in the experiment?

10. Consider the following experiment:

- Start with a pile of 100 kernels of popcorn or dry beans.
- Pour the kernels or beans onto the center of a large paper plate with equal-sized sectors marked as in the following diagram. Shake the plate so that the kernels or beans scatter into the various sectors in a somewhat random pattern.
- Remove all kernels that land on the sectors marked "1" *and* "2" and record the trial number and the number of kernels or beans remaining.
- Repeat the shake-remove-count process several times.



a. If you were to record the results of this experiment in a table of (*trial number, kernels left*) values, what pattern would you expect in that data? What function rule would probably be the best model for the relationship between trial number n and kernels left k?

b. Which of the following data patterns seems most likely to result from performing the experiment and why?

Table I

Trial Number	1	2	3	4	5
Kernels Left	65	40	25	15	10

Table II

Trial Number	1	2	3	4	5
Kernels Left	80	60	40	20	0

Table III

Trial Number	1	2	3	4	5
Kernels Left	35	15	5	2	0

11 Suppose that you performed the following experiment:

- Roll 100 fair dice and remove all that show 2, 4, or 6 dots on the top face.
- Roll the remaining dice and remove all that show 2, 4, or 6 dots on the top face.
- Repeat the roll-and-remove process, recording the number of dice left at each roll.

a. Complete a table like this showing your prediction of the number of dice remaining after each roll-and-remove stage of the experiment.

Roll Number	0	1	2	3	4	5	6	7
Estimated Dice Left	100							

b. Write *NOW-NEXT* and "$y = \ldots$" rules that model the relationship between roll number and dice left shown in your table.

c. Suppose that your teacher claimed to have done a similar experiment, starting with only 30 dice, and got the results shown in the next table.

Roll Number	0	1	2	3	4	5	6
Dice Left	30	17	10	4	3	1	1

Is the teacher's claim reasonable? What evidence supports your judgment?

12 Find values of x and y that will make these equations true statements.

a. $\left(\frac{5}{4}\right)^3 = x$

b. $\left(\frac{5}{d}\right)^2 = \frac{5^x}{d^y}$ $(d \neq 0)$

c. $\left(\frac{n}{4}\right)^3 = \frac{n^x}{y}$

d. $\left(\frac{t^3}{s}\right)^4 = \frac{t^x}{s^y}$ $(s \neq 0)$

13 Write each of the following expressions in a simpler equivalent exponential form.

a. $\left(\frac{4x}{n}\right)^2$

b. $\left(\frac{32x^2y^5}{8x^3y}\right)^2$

c. $\left(\frac{5x}{4y^3}\right)^0$

14 Find values for x and y that will make these equations true statements.

a. $\frac{5^7}{5^5} = 5^y$ **b.** $\frac{3^x}{3^5} = 3^6$ **c.** $\frac{t^5}{t^2} = t^y$ **d.** $\frac{6.4^9}{6.4^9} = 6.4^y$

15 Write each of the following expressions in a simpler equivalent exponential form.

a. $\frac{7^{11}}{7^4}$ **b.** $\frac{25x^3}{5x}$ **c.** $\frac{30x^3y^2}{6xy}$ **d.** $\frac{a^3b^4}{ab^4}$

16 Write each of the following expressions in equivalent exponential form. For those involving negative exponents, write an equivalent form without using negative exponents. For those involving positive exponents, write an equivalent form using negative exponents.

a. 4.5^{-2} **b.** $(7x)^{-1}$ **c.** $\left(\frac{2}{5}\right)^{-1}$ **d.** $\left(\frac{1}{5}\right)^{-4}$

e. $5x^{-3}$ **f.** $\left(\frac{2}{5}\right)^{2}$ **g.** $(4ax)^{-2}$ **h.** $\frac{5}{t^3}$

17 In Parts a–h below, write the number in integer or common fraction form, where possible. Where not possible, write an expression in simplest form using radicals.

a. $\sqrt{49}$ **b.** $\sqrt{28}$ **c.** $98^{\frac{1}{2}}$ **d.** $\sqrt{\frac{64}{25}}$

e. $\sqrt{6}\,\sqrt{24}$ **f.** $\sqrt{9 + 16}$ **g.** $\sqrt{\frac{12}{49}}$ **h.** $(\sqrt{49})^2$

18 Answer these questions about the side and diagonal lengths of squares.

a. How long is the diagonal of a square if each side is 12 inches long?

b. How long is each side of a square if the diagonal is $5\sqrt{2}$ inches long?

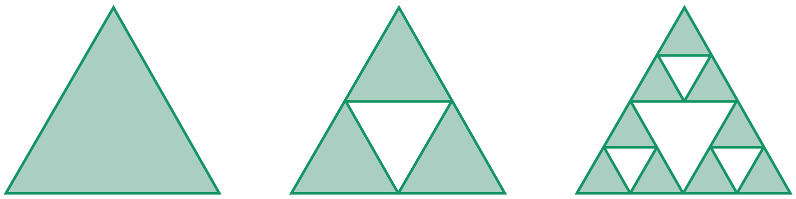
c. How long is each side of a square if the diagonal is 12 inches long?

d. What is the area of a square with a diagonal $5\sqrt{2}$ inches long?

e. What is the area of a square with a diagonal length d units?

Connections

19 One of the most interesting and famous fractal patterns is named after the Polish mathematician Waclaw Sierpinski. The start and first two stages in making a triangular *Sierpinski carpet* are shown below. Assume that the area of the original equilateral triangle is 12 square meters.



Stage 0 Stage 1 Stage 2

a. Sketch the next stage in the pattern. Note how, in typical fractal style, small pieces of the design are similar to the design of the whole.

b. Make a table showing (*cutout stage, area remaining*) data for cutout stages 0 to 5 of this process.

c. Make a plot of the data in Part b.

d. Write two different rules that can be used to calculate the area of the remaining carpet at different stages. One rule should show change from one stage to the next. The other should be in the form "$y = \ldots$."

e. How many stages are required to reach the point where there is:

i. more hole than carpet remaining?

ii. less than 0.1 square meters of carpet remaining?

20 For each of the following rules, decide whether the function represented is an example of:

- An increasing linear function
- A decreasing linear function
- An exponential growth function
- An exponential decay function
- Neither linear nor exponential function

In each case, explain how the form of the rule was used in making your decision.

a. $y = 5(0.4^x)$

b. $y = 5 + 0.4x$

c. $y = 0.4(5^x)$

d. $y = 0.4 + 5x$

e. $y = \frac{5}{x}$

f. $y = \frac{0.4}{x}$

g. $y = 5 - 0.4x$

h. $y = 0.4 - 5x$

i. $NEXT = 0.4 \cdot NOW$

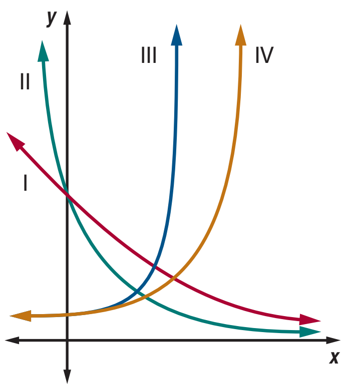
j. $NEXT = NOW + 0.4$

k. $NEXT = NOW - 5$

l. $NEXT = 5 \cdot NOW$

21 The graphs, tables, and rules below model four exponential growth and decay situations. For each graph, there is a matching table and a matching rule. Use what you know about the patterns of exponential relations to match each graph with its corresponding table and rule. In each case, explain the clues that can be used to match the items without any use of a graphing calculator or computer.

Graphs



Tables

A

x	1	2	3	4
y	40	16	6.4	2.56

B

x	1	2	3	4
y	30	90	270	810

C

x	1	2	3	4
y	60	36	21.6	12.96

D

x	1	2	3	4
y	20	40	80	160

Rules

(1) $y = 100(0.6^x)$

(2) $y = 100(0.4^x)$

(3) $y = 10(2^x)$

(4) $y = 10(3^x)$

22 When very large numbers are used in scientific work, they are usually written in what is called *scientific notation*—that is, as the product of a decimal between 1 and 10 (usually rounded to three decimal places) with some power of 10. For example, basic measurements of the Earth are often given in scientific notation like this:

Measurement	Standard Form	Scientific Notation
Land Area (in m^2)	58,969,045,000,000	5.897×10^{13}
Volume (in km^3)	1,083,000,000,000,000,000	1.083×10^{18}
Population	6,214,891,000	6.215×10^{9}
Mass (in kg)	5,976,000,000,000,000,000,000,000	5.976×10^{24}

a. Write each of these large numbers in scientific notation rounded to three decimal places.

i. 234,567,890 **ii.** 54,987 **iii.** 1,024,456,981,876

b. Use negative exponents to write each of these numbers in scientific notation.

i. 0.0234 **ii.** 0.00002056 **iii.** 0.000000000008

c. Translate each of these numbers, given in scientific notation, to standard numeral form.

i. 7.82×10^8 **ii.** 5.032×10^6 **iii.** 8.1×10^{-3}

d. Express the results of these calculations in scientific notation, without using a calculator. Be prepared to explain your reasoning and how you use properties of exponents to reach the results.

i. $(4 \times 10^{12}) \times (3 \times 10^5)$

ii. $(40 \times 10^{12}) \div (5 \times 10^5)$

iii. $(4 \times 10^{12}) \times (3 \times 10^{-5})$

e. Use the Earth measurement data in the table to answer these questions. Express your answers in both scientific and standard notation.

i. How much land surface is there for each person living today?

ii. Each kilogram of mass is equal to 1,000 grams. What is the mass of the Earth in grams?

23 In 2001, the U.S. national public debt was 5.807×10^{12} dollars, and the U.S. population was about 2.85×10^8 people. What does this imply in terms of national public debt per person?

24 Every non-negative number x, has a non-negative square root $\sqrt{x}$.

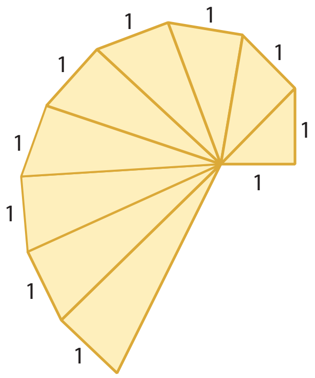
a. Use your calculator to complete the following table of approximate values for the square root function $y = \sqrt{x}$ for whole numbers from 0 to 10. Then sketch a graph of the function.

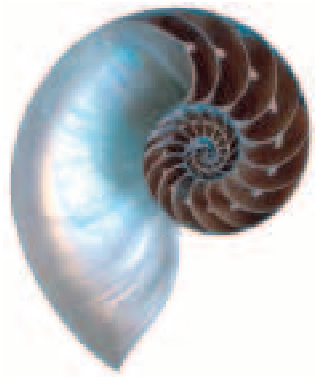
x	0	1	2	3	4	5	6	7	8	9	10
$y = \sqrt{x}$											

b. How is the pattern of change shown in the table and graph of the square root function similar to and different from those of exponential growth and decay functions? How about linear functions?

25 The shell of the chambered nautilus is one of the most beautiful designs in nature. The outside image is a spiral, and segments of the spiral match chambers within the shell that increase in size as the spiral unfolds.

The spiral diagram to the left of the nautilus picture below is similar to the shell. Each outside segment is 1 centimeter long. The individual "chambers" are right triangles.





a. Make a table showing the pattern of lengths for the segments that divide the "shell chambers." Report each segment length in radical form.

b. What rule tells the length of the hypotenuse in the *n*th "chamber"?

Reflections

26 When some students were discussing the ball bounce experiment, one said that he thought the ball might rebound less on each bounce, but it would never actually stop bouncing just a little bit. His partners disagreed. They said that because the rebound height decreases on successive bounces, the rebound *time* also decreases. They said that the sum of rebound times would be:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

What do you think?

On Your Own

27 Suppose a person taking steroid drugs is hospitalized due to a side effect from the drug. Tests taken upon admittance show a steroid concentration of 1.0. The next test one day later shows a concentration of 0.75. Based on these results, the person's family and friends assume that in three more days the drug will be out of the person's system.

a. What pattern of change are the family and friends assuming?

b. What might be a more accurate pattern prediction? Why is that pattern more reasonable?

28 For a function with rule $y = a(b^x)$ where $a > 0$, what conclusions can you draw about the tables and the graphs of (x, y) values when b is

a. between 0 and 1?

b. greater than 1?

29 The definition $b^0 = 1$ is often hard for people to accept. They argue that if b^5 means "5 factors of b," then b^0 should mean "zero factors of b" and this should be zero. The mathematician's response is that sometimes we make definitions for special cases so that they fit together with patterns covering all other cases. For example, they point out that we define $(-3)(-3) = 9$ because

$$\begin{aligned} 0 &= (-3)(0) \\ &= (-3)(3 + (-3)) \\ &= (-3)(3) + (-3)(-3) \\ &= -9 + (-3)(-3) \end{aligned}$$

a. What do you think of this argument for $(-3)(-3) = 9$?

b. What do you think of the general practice of making definitions in special cases so that they fit the rules that apply to all other cases?

30 A student at Sam Houston High School in Texas wrote the following equations to show equivalence of exponential forms. Decide which of the statements are correct. Correct those that are incorrect by revising the right side of the given equations. Give the student advice for correct thinking about the indicated calculations.

a. $\frac{x^2}{x^3} = x$

b. $(2x)(x) = 3x^2$

c. $\frac{30x^3y^3}{6xy} = 24x^3y^3$

Extensions

31 The African Black Rhinoceros is the second largest of all land mammals. The black rhino has walked the earth for 40 million years, and prior to the 19th century over 1,000,000 of the species roamed the plains of Africa. However, that number has been drastically reduced by hunting and loss of natural habitat.

The next table shows the very sharp decline in black rhino numbers between 1970 and 1993.

Year	1970	1980	1984	1986	1993
Population (in 1,000s)	65	15	9	3.8	2.3

Source: *Mammals of the World*, fifth ed., vol. 2. Johns Hopkins University Press: Baltimore, 1991; www.rhinos-irf.org/information/blackrhino/index.htm

a. Experiment with exponential and linear models for the data pattern shown in the table (use 0 for 1970). Decide on a model that seems to be a good fit for the data pattern.

b. Use your model of choice to predict the black rhino population for 2000 and 2005.

c. Since 1996, intense anti-poaching efforts have had encouraging results. Black rhino population estimates for 2000 rose to 2,700 and for 2005 rose to 3,600. Include this additional data and experiment with possible models for the data pattern. Use your model of choice to predict the black rhino population for 2010.

d. Suppose that black rhinos are not poached and their natural habitat is left intact. Assume also that the population would increase at a natural rate of 4% each year after 1993. How would the African population change by 2010 under those conditions?

e. As is the case with populations of Alaskan bowhead whales, native Africans might be allowed an annual hunting quota. Suppose that in 1993, the quota was set at 50 per year. What *NOW-NEXT* rule shows how to explore the effect of this hunting and a 4% natural population growth rate? What 2010 population is predicted under those conditions?

f. Construct a spreadsheet to explore the effects of different natural growth rates and hunting quotas and summarize what you learn in a report. In particular, find hunting quotas that would lead to stable black rhino populations if natural population growth rates were 2%, 5%, 7%, and 10% and the current population is 4,000.

32 Cigarette smoke contains nicotine, an addictive and harmful chemical that affects the brain, nervous system, and lungs. It leads to very high annual health care costs for our country.

Suppose an individual smokes one cigarette every 40 minutes over a period of 6 hours and that each cigarette introduces 100 units of nicotine into the bloodstream. The half-life of nicotine is 20 minutes.

a. Make a chart that tracks the amount of nicotine in that smoker's body over the 6-hour period, making entries in the chart for every 20-minute period. Describe the pattern of nicotine build-up.

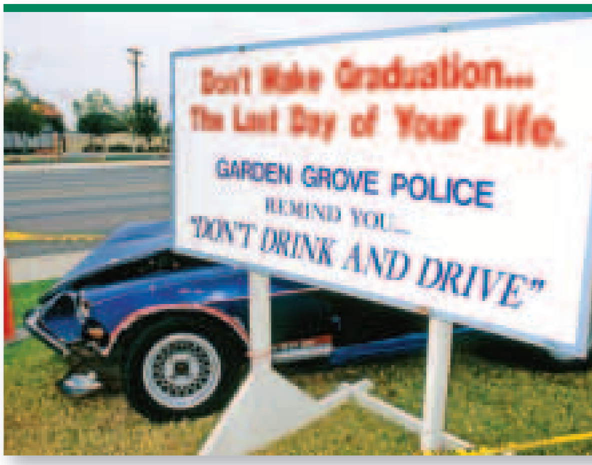
b. Compare the pattern in Part a with the pattern resulting from smoking a cigarette every 20 minutes.

c. Write *NOW-NEXT* rules showing how the amount of nicotine in the body changes over time (in 40-minute intervals for Part a and 20-minute intervals for Part b). Compare these rules to those of simple exponential growth and decay and explain the differences.

d. Because nicotine is strongly addictive, it is difficult for smokers to break the habit. Suppose that a long-time smoker decides to quit "cold turkey." That is, rather than reducing the number of cigarettes smoked, the smoker resolves never to pick up another cigarette. How would the level of nicotine in that smoker's body change over time?

e. How do the results of your analysis in Part d suggest that addiction to nicotine is psychological as well as physical?

33 Driving after drinking alcohol is both dangerous and illegal. The National Highway Traffic Safety Administration reported 2,400 youth (15 to 20 years old) alcohol-related traffic fatalities in 2002—an average of about 6 per day. (Source: *Traffic Safety Facts 2002: Young Drivers*; www.nhtsa.dot.gov)



Many factors affect a person's Blood Alcohol Concentration (BAC), including body weight, gender, and amount drunk. American Medical Association guidelines suggest that a BAC of 0.05 is the maximum safe level for activities like driving a car.

The following chart gives typical data relating body weight and number of drinks consumed to BAC for people of various weights.

Approximate Blood Alcohol Concentrations

Weight (in pounds)	1 drink	2 drinks	3 drinks	4 drinks	5 drinks
100	0.05	0.09	0.14	0.18	0.23
120	0.04	0.08	0.11	0.15	0.19
140	0.03	0.07	0.10	0.13	0.16
160	0.03	0.06	0.09	0.11	0.14
180	0.03	0.05	0.08	0.10	0.13

a. Study the data in the table and decide how BAC for each weight seems to be related to number of drinks consumed. Find *NOW-NEXT* and "$y = \ldots$" rules for the function that seems the best model at each weight. Explain what each rule tells about the effects of additional drinks on BAC.

b. The next table shows how BAC changes over time after drinking stops for a 100-pound person who has had 3 drinks.

Time (in hours)	0	2	4	6	8
BAC	0.14	0.12	0.10	0.08	0.06

i. What type of function seems to model that pattern of change well?

ii. Find *NOW-NEXT* and "$y = \ldots$" rules for the function that seems the best model.

iii. Explain what those rules tell about the way BAC declines over time.

CPMP-Tools

c. Suppose that the pattern in Part b relating BAC to time since last drink applies for the situation in which a 100-pound person has 5 drinks. What is the prediction of time required for the blood alcohol of that 100-pound person to return to a “safe” level of 0.05?

34 To study behavior of exponential functions for fractional values of the independent variable, consider several numbers between 0 and 1, like 0.25, 0.5, and 0.75.

a. How would you expect the values of 5^0, $5^{0.25}$, $5^{0.5}$, $5^{0.75}$, and 5^1 to be related to each other?

b. What are your best estimates for the values of $5^{0.25}$, $5^{0.5}$, and $5^{0.75}$?

c. Use a calculator to check your ideas in Parts a and b.

d. Graph the function $y = 5^x$ for $-1 \leq x \leq 2$ and explain how it shows the observed pattern of change in values for 5^0, $5^{0.25}$, $5^{0.5}$, $5^{0.75}$, and 5^1.

35 Use what you know about properties of exponents to evaluate these expressions.

i. $\left(3^{\frac{1}{4}}\right)^4$	**ii.** $\left(5^{\frac{1}{4}}\right)^4$	**iii.** $\left(16^{\frac{1}{4}}\right)^4$
iv. $16^{\frac{1}{4}}$	**v.** $\left(16^{\frac{1}{4}}\right)^3$	**vi.** $(16)^{\frac{3}{4}}$

a. Look for a pattern to help you explain what $b^{\frac{1}{4}}$ and $b^{\frac{3}{4}}$ must mean for any positive value of b.

b. Based on your explorations, what meanings are suggested for the expressions $b^{\frac{1}{n}}$ and $b^{\frac{m}{n}}$ when b is a positive number and m and n are positive integers? Use your calculator to check your ideas in the case of some specific examples.

36 Any number that can be expressed as an integer or a common fraction is called a *rational number*. For example, 12, $\frac{3}{5}$, and $\frac{7}{5}$ are rational numbers. If a number cannot be expressed as an integer or common fraction, it is called an *irrational number*.

When you use a computer algebra system for arithmetic and algebraic calculations, it will generally report any numerical results as integers or common fractions whenever that is possible.

a. Use a computer algebra system to see which of the numbers $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots, \sqrt{50}$ are rational numbers and which are irrational.

b. Based on your work in Part a, what seems to be a way to decide when $\sqrt{n}$ is rational and when it is irrational?

c. Use a computer algebra system to evaluate $\sqrt{\frac{a}{b}}$ for several different fractions like $\sqrt{\frac{4}{9}}, \sqrt{\frac{4}{7}}, \sqrt{\frac{5}{9}}$, and $\sqrt{\frac{6}{35}}$.

d. Based on your work in Part c, what seems to be a way to decide when $\sqrt{\frac{a}{b}}$ is rational and when it is irrational?

Review

37 Work on problems that involve exponential growth and decay often requires skill in use of percents to express rates of increase or decrease. Suppose that you are asked to figure new prices for items in a sporting goods store. Show two ways to calculate each of the following price changes—one that involves two operations (either a multiplication and an addition or a multiplication and a subtraction) and another that involves only one operation (multiplication).

a. Reduce the price of a \$90 warm-up suit by 20%.

b. Increase the price of a \$25 basketball by 30%.

c. Reduce the price of a \$75 skateboard by 60%.

d. Increase the price of a \$29 sweatshirt by 15%.

e. Reduce the price of a \$15 baseball cap by $33\frac{1}{3}$%.

f. Increase the price of a \$60 tennis racket by 100%.

38 Sketch and label a diagram for each situation. Then find the measure of the indicated segment or angle.

a. $m\angle XYZ = 140°$ and $\overrightarrow{YB}$ bisects $\angle XYZ$. Find $m\angle BYZ$.

b. $AB = 5$ cm and C is the midpoint of $\overline{AB}$. Find AC.

c. M is the midpoint of $\overline{XY}$, $XM = 3$ cm. Find XY.

d. $\overrightarrow{PQ}$ bisects $\angle RPT$. $m\angle RPQ + m\angle QPT = 82°$. Find $m\angle RPT$ and $m\angle RPQ$.

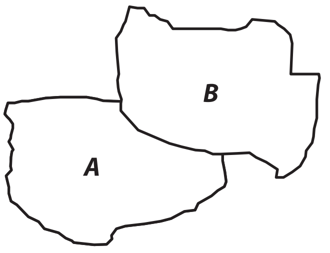
39 Write *NOW-NEXT* rules that match each of the following *linear decay* functions. Then explain how the "decay" in all three cases is different from exponential decay and how the difference(s) would appear in tables and graphs of (x, y) values for the functions.

a. $y = 5 - 2x$

b. $y = -0.5x + 1$

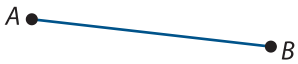
c. $y = -\frac{x}{3} - 6$

40 In Unit 4, *Vertex-Edge Graphs*, you learned about coloring maps and coloring vertex-edge graphs. Suppose the sketch at the right represents a simple map, where country A shares a border with country B. (Rules forbid having countries meet at a point.)



a. Copy this map. Add a country C that shares a border with both countries A and B. Now add a country that shares a border with each of countries A, B, and C. Continue to add a country, one at a time, each country sharing a border with *each* of the previous countries. Try different placements of the countries. What was the maximum number of countries possible?

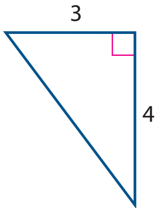
b. The vertex-edge graph shown represents the same situation as the map on the previous page—the edge represents a shared border. As before, add vertices (countries) and edges (borders) one at a time. What is the maximum number of vertices on the graph?



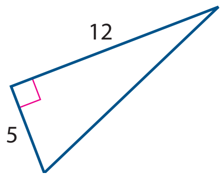
c. Conjecture: If every country shares a border with every other country, then the largest number of countries possible is __________ .

41 Find the missing side lengths in each triangle. Express your answers in both radical and decimal approximation form.

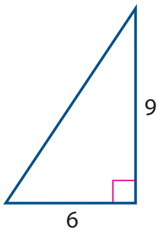
a.



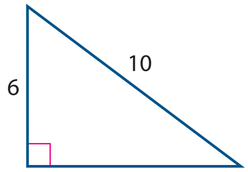
b.



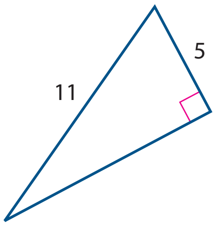
c.



d.



e.



42 Without using a calculator, decide if the following statements are true or false. If the statement is false, explain why.

a. $2 \cdot 2^5 = 4^5$

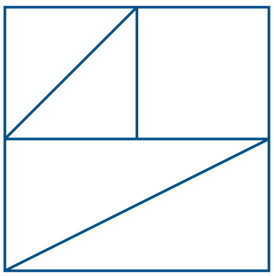
b. $2 \cdot 2^5 = 2^5$

c. $2 \cdot 2^5 = 2^6$

d. $2 \cdot 3^x + 3^x = (2 + 1)3^x$

e. $3 \cdot 3^x = 3^{x + 1}$

43 Make a copy of the diagram at the right. Shade the part of the large square that represents $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$, and give the answer as a fraction.



44 Examine each equation below and decide on the method you would use to solve it: use a table, use a graph, or reason with the symbols themselves. Then solve each equation using your preferred method and check your answer.

a. $2^x = 100$

b. $2x + 5 = 100$

c. $100(x + 2) = 800$

d. $100(1.5^x) = 200$

e. $100(b^4) = 200$

45 Write an equation for the line that matches each description.

a. Has slope of −0.5 and y-intercept at (0, 2)

b. Contains the points (0, 5) and (−4, −10)

c. Is horizontal and contains the point (7, 12)

d. Contains the points (−2, 7.5) and (1, 3)

46 A basketball team is selling sweatshirts in order to raise money for new uniforms. The rule that gives their profit p in dollars based on the number of sweatshirts they sell n is $p = 5n - 175$.

a. Explain the meaning of the 175 and the 5 in terms of the situation.

b. If they sell 265 sweatshirts, how much profit will they make?

c. How many sweatshirts must they sell in order to make a profit of $2,000?

LESSON 3

Looking Back

In this unit, you studied patterns of change in variables that can be modeled well by exponential functions. These functions can all be expressed with rules like

$$NEXT = b \cdot NOW, \text{ starting at } a$$
or
$$y = a(b^x).$$

As a result of your work on Lessons 1 and 2, you should be better able to recognize situations in which variables are related by exponential functions, to use data tables and graphs to display patterns in those relationships, to use symbolic rules to describe and reason about the patterns, and to use graphing calculators, spreadsheets, and computer algebra systems to answer questions that involve exponential relationships. You should also be able to write exponential expressions in useful equivalent forms.

The tasks in this final lesson will help you review your understanding of exponential functions and apply that understanding in solving several new problems.

CPMP-Tools

1 **Counting Codes** Code numbers are used in hundreds of ways every day—from student and social security numbers to product codes in stores and membership numbers in clubs.

a. How many different 2-digit codes can be created using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 (for example, 33, 54, 72, or 02)?

b. How many different 3-digit codes can be created using those digits?

c. How many different 4-digit codes can be created using those digits?

d. Using any patterns you may see, complete a table like the one below showing the relation between number of digits and number of different possible codes.

Number of Digits	1	2	3	4	5	6	7	8	9
Number of Codes									

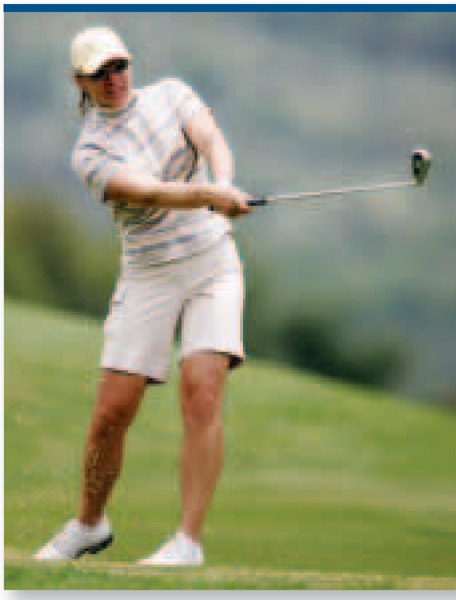
e. Write a rule using *NOW* and *NEXT* to describe the pattern in the table of Part d.

f. Write a rule that shows how to calculate the number of codes C for any number of digits D used.

g. Music and video stores stock thousands of different items. How many digits would you need in order to have code numbers for up to 8,500 different items?

h. How will your answers to Parts d–f change if the codes were to begin with a single letter of the alphabet (A, B, C, ... , or Z) as in A23 or S75?

2 **Eyes on the Prizes** In one women's professional golf tournament, the money a player wins depends on her finishing place in the standings. The first-place finisher wins $\frac{1}{2}$ of the $1,048,576 in total prize money. The second-place finisher wins $\frac{1}{2}$ of what is left; then the third-place finisher wins $\frac{1}{2}$ of what is left, and so on.



U.S. Women's Open Champion Annika Sorenstam

a. What fraction of the *total* prize is won

- **i.** by the second-place finisher?
- **ii.** by the third-place finisher?
- **iii.** by the fourth-place finisher?

b. Write a rule showing how to calculate the fraction of the total prize money won by the player finishing in nth place, for any positive integer n.

c. Make a table showing the actual prize money in dollars (not fraction of the total prize money) won by each of the first five place finishers.

Place	1	2	3	4	5
Prize (in dollars)					

d. Write a rule showing how to calculate the actual prize money in dollars won by the player finishing in place n. How much money would be won by the 10th-place finisher?

e. How would your answers to Parts a–d change if

- **i.** the total prize money were reduced to $500,000?
- **ii.** the fraction used was $\frac{1}{4}$ instead of $\frac{1}{2}$?

f. When prize monies are awarded using either fraction $\frac{1}{2}$ or $\frac{1}{4}$, could the tournament organizers end up giving away more than the stated total prize amount? Explain your reasoning.

3 **Cold Surgery** Hypothermia is a life-threatening condition in which body temperature falls well below the norm of 98.6°F. However, because chilling causes normal body functions to slow down, doctors are exploring ways to use hypothermia as a technique for extending time of delicate operations like brain surgery.

a. The following table gives experimental data illustrating the relationship between body temperature and brain activity.

Body Temperature (in °F)	50	59	68	77	86	98.6
Brain Activity (% Normal)	11	16	24	37	52	100

Source: *USA Today*, August 1, 2001, "Surgery's Chilling Future Will Put Fragile Lives on Ice."

i. Plot the table data and find a "$y = \ldots$" rule that models the pattern in these data relating brain activity level to body temperature. Then express the same relationship with an equivalent *NOW-NEXT* rule.

ii. Use your rules to estimate the level of brain activity at a body temperature of 39°F, the lowest temperature used in surgery experiments on pigs, dogs, and baboons.

iii. Find the range of body temperatures at which brain activity is predicted to be about 75% of normal levels.

b. The next table gives experimental data illustrating the relationship between body temperature and safe operating time for brain surgery.

Body Temperature (in °F)	50	59	68	77	86	98.6
Safe Operating Time (in minutes)	45	31	21	14	9	5

i. Plot the table data and find a "$y = \ldots$" rule that models the pattern in these data relating safe operating time to body temperature. Then express the same relationship with an equivalent *NOW-NEXT* rule.

ii. Use your rules to estimate the safe operating time at a body temperature of 39°F.

iii. Find the body temperature at which safe operating time is predicted to be at least 25 minutes.

Roger Ressmeyer

c. Cost is another important variable in medical practice. The next table gives data about charges for a sample of routine surgeries, illustrating the relationship between time required for the operation and hospital charges for use of the operating room.

Time (in minutes)	30	60	90	120	150	180
Cost (in $)	950	1,400	1,850	2,300	2,750	3,200

i. Plot the table data and find a "$y = \ldots$" rule that models the pattern in these data relating surgery cost to time. Then express the same relationship with an equivalent *NOW-NEXT* rule.

ii. Use your rules to estimate the cost of an operation that takes 45 minutes.

iii. Find the time of an operation for which cost is predicted to be $5,000.

d. Compare the "$y = \ldots$" rules for the three functions in Parts a, b, and c. In each case, explain how the rules alone can be used to predict the pattern of change in the dependent variable as the independent variable increases.

4 **Exponent Properties** In Lessons 1 and 2, you discovered and practiced several principles for writing exponential expressions in equivalent (often simpler) forms. Use those principles to find values of x and y that make the following equations true statements.

a. $(2.3^5)(2.3^3) = 2.3^x$

b. $2.3^x = 1$

c. $(3.5^x)^y = 3.5^{12}$

d. $\frac{7^9}{7^4} = 7^x$

e. $\frac{7^x}{7^4} = 7^2$

f. $(7^3)^x = 7^6$

g. $\left(\frac{3}{5}\right)^4 = \frac{3^x}{5^y}$

h. $(4a)^3 = 4^x a^y$

i. $\frac{1}{7^4} = 7^x$

5 **Fractional Powers and Radicals** In Lesson 2, you also discovered and practiced use of expressions in which fractional powers occur. Special attention was paid to square roots, using the exponent one-half. Use what you learned to answer these questions.

a. The value of $3^2 = 9$ and $3^3 = 27$. What does this information tell about the approximate values of $3^{2.4}$ and $3^{2.7}$?

b. For each of these equations, find two different pairs of integer values for a and b that make the equation true.

i. $\sqrt{48} = a\sqrt{b}$

ii. $\sqrt{a}\,\sqrt{b} = \sqrt{36}$

Summarize the Mathematics

When two variables are related by an exponential function, that relationship can be recognized from key features of the problem situations, from patterns in tables and graphs of (x, y) data, and from the rules that show how to calculate values of one variable from given values of the other.

a In deciding whether an exponential function describes the relationship between two variables, what hints do you get from

- **i.** the nature of the situation and the variables involved?
- **ii.** the patterns in graphs or scatterplots?
- **iii.** the patterns in data tables?

b Exponential functions, like linear functions, can be expressed by a rule relating x and y values and by a rule relating *NOW* and *NEXT* y values.

- **i.** Write a general rule for an exponential function, "$y = \ldots$."
- **ii.** Write a general rule relating *NOW* and *NEXT* for an exponential function.
- **iii.** What do the parts of the rules tell you about the problem situation?
- **iv.** How do you decide whether a given exponential function rule will describe growth or decay, and why does your decision rule make sense?

c Suppose that you develop or discover a rule (*NOW-NEXT* or "$y = \ldots$") that shows how a variable y is an exponential function of another variable x. Describe the different strategies you could use to complete tasks like these:

- **i.** Find the value of y associated with a specific given value of x.
- **ii.** Find the value of x that gives a specific target value of y.
- **iii.** Describe the way that the value of y changes as the value of x increases or decreases.

d Complete each equality to give a useful equivalent form of the first expression.

i. $a^m a^n = \ldots$	**ii.** $(a^m)^n = \ldots$	**iii.** $a^0 = \ldots$
iv. $(ab)^n = \ldots$	**v.** $\frac{1}{a^n} = \ldots$	**vi.** $\left(\frac{a}{b}\right)^n = \ldots$
vii. $\frac{a^m}{a^n} = \ldots$	**viii.** $\sqrt{ab} = \ldots$	**ix.** $\sqrt{\frac{a}{b}} = \ldots$

Be prepared to share your responses and thinking with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.