UNIT 1

REASONING AND PROOF

Life presents many opportunities to use logical reasoning in solving problems and drawing conclusions from information. Whether it is developing a winning strategy in a favorite game, figuring out how to build or repair something, or learning how to drive defensively, the ability to ask yourself "What will happen if … ?" is fundamental. Quite often, it is the ability of an athlete, a detective, or a lawyer to use reasoning that makes that person stand out from the crowd.

In this unit, you will examine more carefully the reasoning strategies you have used in your prior mathematics study. You will learn some basic principles of logical reasoning that underline those strategies and develop skill in applying that understanding to mathematical questions in geometry, algebra, and statistics. The key ideas will be developed through work on problems in four lessons.

Lessons

1 *Reasoning Strategies*

Analyze and use deductive and inductive reasoning strategies in everyday situations and in mathematical contexts.

2 *Geometric Reasoning and Proof*

Use inductive reasoning to discover and deductive reasoning to prove relations among angles formed by two intersecting lines or by two parallel lines and a transversal.

3 *Algebraic Reasoning and Proof*

Use symbolic notation to represent numerical patterns and relationships and use rules for transforming algebraic expressions and equations to prove those facts.

4 *Statistical Reasoning*

Know the characteristics of a well-designed experiment and use statistical reasoning to decide whether one treatment causes a better result than a second treatment.

LESSON 1

Reasoning Strategies

In Courses 1 and 2 of *Core-Plus Mathematics*, you frequently used *inductive reasoning* to discover general patterns or principles based on evidence from experiments or several cases. You also used *deductive reasoning* to justify statements or conclusions based on accepted facts and definitions. In this unit, you will expand your ability to reason carefully in geometric, algebraic, and statistical contexts.

Careful reasoning is important not only in mathematics. It is a key to success as a consumer, in careers, and even in recreational activities. Setting up a play in basketball or volleyball, winning a friendly game of CLUE® or MONOPOLY®, or solving a crossword puzzle involves careful strategic reasoning.

Consider the game of Sudoku (pronounced sue-doe-koo), a number game similar to a crossword puzzle, that has recently become popular around the world. Sudoku is a Japanese word meaning "single number." The goal of the game is to fill in the empty squares with the digits 1 to 9 so that each digit appears exactly once in every row, column, and outlined 3×3 block.

8				6				1
4	5	7	8					9
			2	5	9			4
2	3	9		8				
6	8						9	3
	7			9		1	6	8
7			9	2	8			
9			4		5	7	3	
1				7	3			2

Adapted from: www.knightfeatures.com

Think About This Situation

Think about strategies you would use to solve the Sudoku puzzle shown at the bottom of the previous page.

a How would you decide where to begin?

b Which square would you fill in first? Which one would you fill in next? Explain your reasoning.

c Describe a strategy (or a combination of strategies) you would use to fill in the remaining squares.

d When the game is completed, what will be true about the sums of the row entries, the sums of the column entries, and the sums of the 3×3 block entries? Explain.

In this lesson, you will learn how to examine arguments in terms of reasoning strategies, assumptions, and logical soundness. You will also learn how to use *if-then* reasoning patterns in deductive arguments or proofs.

Investigation 1 Reasoned Arguments

Careful reasoning, whether it is concerned with mathematics, science, history, or daily affairs, is important if you want to have confidence in the conclusions reached. A *valid argument* shows that a conclusion follows logically from accepted definitions and assumptions or previously established facts. If the assumptions are true, you can be confident that the conclusion is true. As you analyze the situations and arguments in this investigation, look for answers to this question:

How can you determine whether a conclusion follows logically from information and facts you know are correct or on which everyone would agree?

1 **Reasoning about Crime Scenes**
In popular television shows like CSI and NUMB3RS that involve crime investigations, the detectives use careful reasoning to identify suspects, motives, and evidence that can be used to solve cases.

Consider the following plot from a crime case.

> At 7:00 P.M., Mrs. Wilson's maid served her tea in the library. The maid noticed that Mrs. Wilson seemed upset and a little depressed. At 8:45 P.M., the maid knocked on the library door and got no answer. The door was locked from the inside. The maid called Inspector Sharpe and a professor friend. When the door was forced open, Mrs. Wilson was found dead. The maid burst into tears, crying, "I feel so bad that we haven't been getting along lately!" Nearby was a half-empty teacup, a tiny unstoppered vial, and a typewritten note that said, "Blessed are the poor for they shall be happy." The window was open. When the two men went out-side to inspect the grounds, Charles, the wealthy widow's sole heir, arrived. He was told his aunt was poisoned and said, "How terrible! Poisoned? Who did it? Why was the door locked? Had my aunt been threatened?" He explained he had been working late at the office and was stopping by on his way home.

Source: Ripley, Austin. (1976). *Minute Mysteries.* New York: Harper & Row Publishers. pages 22–23.

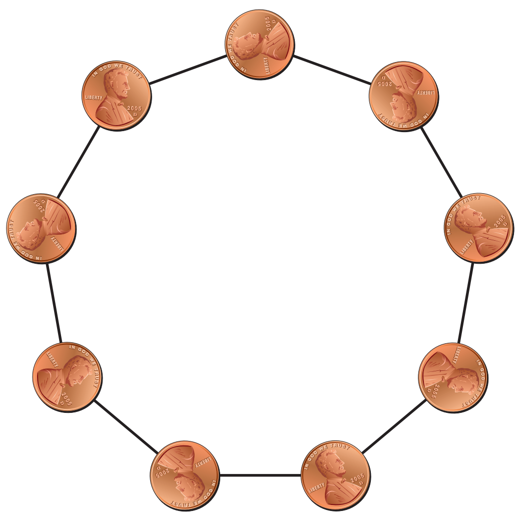
a. Who are the possible suspects in this case?

b. For each suspect, identify the evidence that exists that could be used to charge them with the crime.

c. Write a convincing argument to charge the prime suspect with the crime.

d. Compare your prime suspect and argument with those of others. Resolve any differences.

2 **Reasoning about Games** Games based on strategy—such as Tic-Tac-Toe, Checkers, and Chess—have been played for thousands of years. The following two-person game can be played on any regular polygon. For this problem, assume the game is played on a regular nonagon. To play, place a penny on each vertex of the polygon. Take turns removing one penny or two pennies from adjacent vertices. The player who picks up the last coin(s) is the winner.



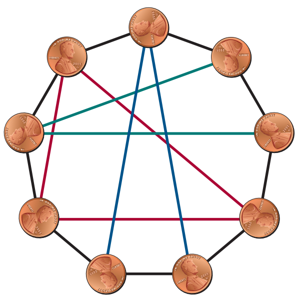
a. Working with a partner, play the nonagon game a few times. Make mental notes of strategies you used.

b. Tianna, Jairo, Nicole, and Connor each thought they found a strategy that would guarantee that they could always win the game if they played *second*. Analyze their reasoning. In each case, decide if the proposed strategy will always result in a win for the second player. Explain your reasoning in each case.

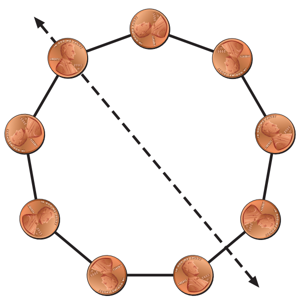
Tianna: Each time the first player removes one or two pennies, I'll remove only one penny. That way, there will be more pennies left to choose from and at least one for me at the end.

Jairo: Each time the first player removes one penny, I'll remove one penny. If the first player removes two pennies, then I'll remove two pennies. Then the remaining number of pennies will be an odd number, so there will be at least one penny for me at the end.

Nicole: Each time the first player removes one penny, I'll remove two pennies; if the first player removes two pennies, I'll remove one penny as shown by the triangles in the diagram below. Since there are 9 pennies, I can always win on the third round.



Connor: If the first player removes one penny, I'll visualize the line of symmetry containing the "empty" vertex and remove the two adjacent pennies on opposite sides of the symmetry line. If the first player removes two pennies on the first move, then I'll visualize the line of symmetry between the two empty vertices and remove the penny on the symmetry line.



This strategy will always leave three pennies on each side of the symmetry line after we each make our first move. After that, I'll match each play the first player makes by choosing the mirror image. So, I'll be able to remove the final coin(s).

3 **Reasoning about Numbers** You may have noticed that when you add two odd numbers, for example 3 + 7, the sum always seems to be an even number. Alex, Maria, Nesrin, and Teresa were asked to write an argument to justify the following claim.

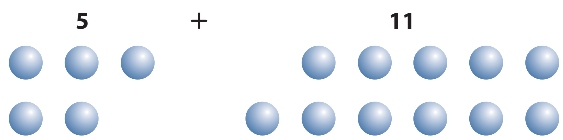
If a and b are odd numbers, then a + b (the sum) is an even number.

Carefully study each student's argument.

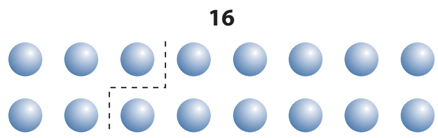
Alex: I entered odd numbers in list L_1 of my calculator and different odd numbers in list L_2. I then calculated $L_1 + L_2$. Scanning the calculator screen, you can see that in every case the sum is an even number.

Maria: Odd numbers end in 1, 3, 5, 7, or 9. When you add any two of these, the answer will end in 0, 2, 4, 6, or 8. So, the sum of two odd numbers must always be an even number.

Nesrin: I can use counters to prove the sum of any two odd numbers is an even number. For example, if I take the numbers 5 and 11 and organize the counters as shown, you can see the pattern.



You can see that when you put the sets together (add the numbers), the two extra counters will form a pair and the answer is always an even number.



Teresa: I know that a and b are odd numbers.
By definition, $a = 2m + 1$ and $b = 2n + 1$, where m and n are integers.
So, $a + b = 2m + 1 + 2n + 1$.
Then $a + b = 2m + 1 + 2n + 1 = 2m + 2n + 2 = 2(m + n + 1)$.
Therefore, $a + b$ is an even number since the sum is a multiple of 2.

a. Of the four arguments, which *one* is closest to the argument you would give to prove that the sum of two odd numbers is an even number?

b. For each of the arguments, answer the following questions.

i. Does the argument have any errors in it?

ii. Does the argument show the statement is *always true* or does the argument *only* show the statement is true for *some* numbers?

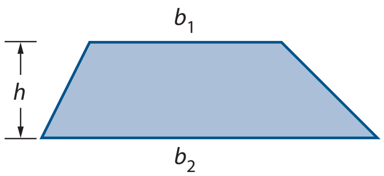
iii. Does the argument show *why* the statement is true?

iv. Does the argument provide an easy way to convince someone in your class who is uncertain of the claim?

c. Select one of the arguments you think is correct. How, if at all, would you modify the argument to justify that the sum of *any* two odd numbers that are square numbers (like 9 and 25) is an even number? Explain your reasoning.

4 **Reasoning about Areas** In the Course 1 *Patterns in Shape* unit, you saw that by assuming the formula $A = bh$ for the area of a rectangle with a base of length b and height h, you could derive a formula for the area of a parallelogram. You also saw that if you knew the formula for the area of a parallelogram, you could derive a formula for the area of a triangle. A standard formula for calculating the area of a **trapezoid**—a quadrilateral with two opposite sides parallel—is given by:

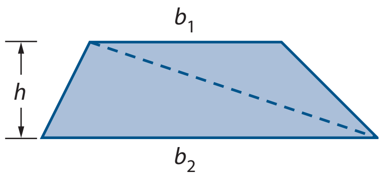
$$A = \frac{1}{2}(b_1 + b_2)h$$



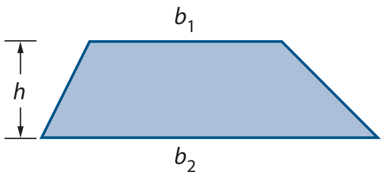
where b_1, b_2, and h represent the lengths of the two bases and the height of the trapezoid.

Study each of the following five arguments offered by students as justification of this formula for the area of a trapezoid.

Angela: I can split the trapezoid into two triangles by drawing a diagonal. One triangle has area $\frac{1}{2}b_1h$. The other has area $\frac{1}{2}b_2h$. So, the area of the trapezoid is $\frac{1}{2}b_1h + \frac{1}{2}b_2h$ or $\frac{1}{2}(b_1 + b_2)h$.



Dylan: In a parallelogram, opposite sides are the same length. Any side can be used as the base. In the trapezoid shown,



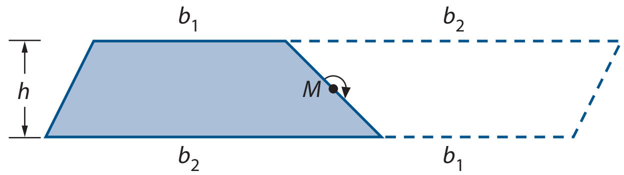
b_1h will underestimate the area.

b_2h will overestimate the area.

To find the correct area, you average the two estimates.

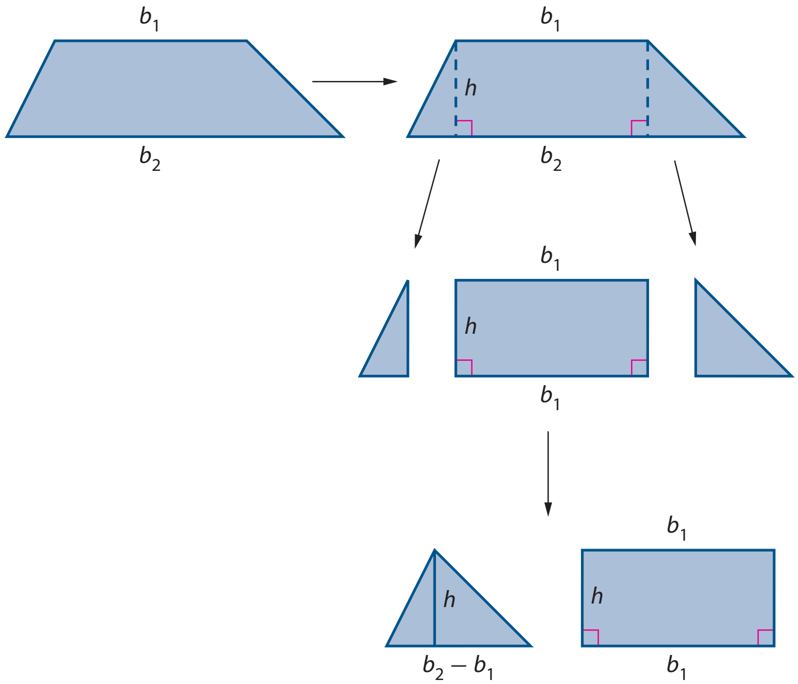
$$\frac{b_1h + b_2h}{2} = \frac{1}{2}(b_1h + b_2h)$$
$$= \frac{1}{2}(b_1 + b_2)h$$

Hsui: If I rotate the trapezoid 180° about the midpoint M of one side, the trapezoid and its image form a parallelogram.



The length of the base of the parallelogram is $b_2 + b_1$ and the height is h. The area of the parallelogram is $(b_2 + b_1)h$. The area of the trapezoid is $\frac{1}{2}$ of this area, or $\frac{1}{2}(b_1 + b_2)h$.

Barbara:



So, the area of the trapezoid is $\frac{1}{2}(b_2 - b_1)h + b_1h$, which equals $\frac{1}{2}(b_1 + b_2)h$.

Jorge: The area of the trapezoid is $\frac{1}{2}(b_1 + b_2)h$ because you can cut up the shape and find the areas of the individual pieces.

a. Which of the five arguments is closest to the argument you would have provided to justify the formula for the area of a trapezoid?

b. Which of these arguments show correct reasoning and which do not? Compare your responses with those of others and resolve any differences.

c. Select one of the arguments you think provides a correct proof of the area formula. Describe the features of the argument that you thought were good. What, if anything, would you add to that argument to make it easier to understand?

Summarize the Mathematics

In this investigation, you examined reasoning strategies and arguments in mathematical and nonmathematical contexts.

a Look back at the mathematical statements that the students were attempting to prove in Problems 3 and 4. In each case, answer the following questions.

i. What information was given?

ii. What conclusion was to be established?

iii. How was the given information used by Teresa to reason logically to the conclusion in Problem 3? By Angela to reason to the conclusion in Problem 4?

b How can you tell whether an argument provides a correct proof of a claim?

Be prepared to share your ideas with the rest of the class.

Check Your Understanding

Analyze each student attempt to prove the following statement.

The sum of two even numbers is an even number.

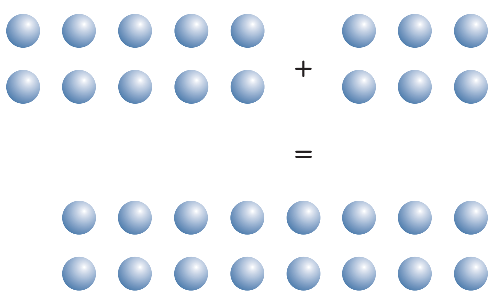
Art: I tried many different pairs of even numbers. It was impossible to find a counterexample to the claim that the sum of two even numbers is an even number. So, the claim must be true.

Sherita: Even numbers are numbers that can be divided by 2. When you add numbers with a common factor of 2, the answer will have a common factor of 2.

Bill: If a and b are any two numbers, then $2a$ and $2b$ are any two even numbers.

$2a + 2b = 2(a + b)$

Katrina: Even numbers can be represented by rectangular arrays of counters with two rows.



The sum of any two even numbers will be a rectangular array of counters with 2 rows. So, the sum is even.

a. Which argument is closest to the argument you would give to prove that the sum of two even numbers is always an even number?

b. Which arguments are *not* correct proofs of the claim? Explain your reasoning.

c. Modify one of the correct proofs to justify the following statement.

The sum of an even number and an odd number
is always an odd number.

Investigation 2 Reasoning with If-Then Statements

Statements of the form "*If* … , *then* …" occur frequently in everyday life and in mathematics. For example, consider the two statements: "*If* it is raining on game day, *then* the game will be rescheduled for next Tuesday," and "*If* $x > 5$, *then* $2x > 10$." Other mathematical statements, such as definitions, can be interpreted in *if-then* form. For example, consider the definition of a trapezoid from Investigation 1.

A trapezoid is a quadrilateral with two opposite sides parallel.

This definition means that *If* a quadrilateral is a trapezoid, *then* two opposite sides are parallel,

and the *converse* *If* two opposite sides of a quadrilateral are parallel, *then* the quadrilateral is a trapezoid.

If-then statements are frequently used in deductive arguments because they imply that if some condition (called the **hypothesis**) is satisfied, some other condition (called the **conclusion**) follows. As you work on the following problems, look for answers to these questions:

How can you use if-then statements in deductive reasoning?

How is deductive reasoning with if-then statements
different from inductive reasoning from patterns?

1 You may recall that the numbers 2, 3, 7, and 11 are called prime numbers. By definition, a **prime number** is an integer greater than 1 that has exactly two factors, 1 and itself.

a. Write two if-then statements that together mean the same thing as the above definition of a prime number.

b. Is 23 a prime number? Explain your reasoning.

c. Which of the two if-then statements in Part a was used in your reasoning in Part b?

The reasoning you used in Problem 1 Part b is based on a fundamental principle of logic called **modus ponens** (Latin: mode that affirms) or **Affirming the Hypothesis**.

If you have a known fact an (if-then) statement that is always true,
and you also know the "if" part is true in a particular case,
you can conclude the "then" part is true in that case.

2 Decide what can be concluded, if anything, from each of the following sets of statements. Be prepared to explain how reaching your conclusion involved Affirming the Hypothesis.

a. *Known fact:* *If* a person has a Michigan driver's license, *then* the person is 16 years of age or older.

Given: Andy has a Michigan driver's license.

Conclusion: ?

b. *Known fact:* *If* a person in Michigan has a driver's license, *then* the person is 16 years of age or older.

Given: Janet is 18 years old.

Conclusion: ?

c. *Known fact:* *If* two sides of a triangle are the same length, *then* the triangle is isosceles.

Given: $\triangle ABC$ has sides of length 2 cm, 5 cm, 5 cm.

Conclusion: ?

d. *Known fact:* *If* $f(x) = ax^2 + bx + c$ is a quadratic function with $a < 0$, *then* $f(x)$ has a maximum value.

Given: $g(x) = -8x^2 + 5x - 2$

Conclusion: ?

e. *Known fact:* *If* a connected vertex-edge graph has vertices all of even degree, *then* the graph has an Euler circuit.

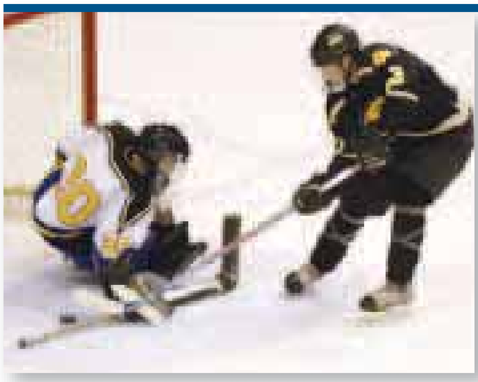
Given: G is a connected vertex-edge graph with 6 vertices.

Conclusion: ?

f. *Known fact:* *If* a data set with mean $\bar{x}$ and standard deviation s is transformed by adding a constant c to each value, *then* the mean of the transformed data set is $\bar{x} + c$ and the standard deviation is s.

Given: The Oak Park hockey team has a mean height of 5 feet 9 inches and a standard deviation of $2\frac{1}{2}$ inches. Wearing ice skates adds approximately $1\frac{3}{4}$ inches to the height of a skater.

Conclusion: ?



g. *Known fact:* If S is a size transformation with center at the origin and magnitude k and A is the area of a figure, then $k^2 \cdot A$ is the area of the image of the figure under S.

Given: $\triangle P'Q'R'$ is the image of $\triangle PQR$ under a size transformation with center at the origin and magnitude 3. $\triangle PQR$ has area 32 cm^2.

Conclusion: ?

3 If-then statements can be represented symbolically as $p \Rightarrow q$ (read "if p, then q" or "p implies q") where p represents the hypothesis and q represents the conclusion. The arrow signals that you move from the hypothesis p to the conclusion q. The reasoning pattern you used in Problem 2 (Affirming the Hypothesis) can be represented as follows.

	Words	Symbolic Form
Known fact:	"If p, then q" is always true,	$p \Rightarrow q$
Given:	**and** p is true in a particular case,	$\underline{p \quad\quad}$
Conclusion:	**then** q is true in that case.	q

In the symbolic form, everything above the horizontal line is assumed to be correct or true. What is written below the line follows logically from the accepted information.

a. In Problem 2 Part c, identify p and q in the general statement $p \Rightarrow q$. Identify the specific case of p. Of q.

b. In Problem 2 Part d, identify p and q in the general statement $p \Rightarrow q$. Identify the specific case of p. Of q.

The "known facts" used in Problem 2 Parts c–g are definitions or principles and relationships you discovered and, as a class, agreed upon in previous mathematics courses. Your discoveries were probably based on studying several particular cases or conducting experiments and then searching for patterns.

4 Select one of the statements given as a known fact in Problem 2 Parts d–g. Discuss with classmates how you could explore specific cases that might lead to a discovery of a pattern suggesting the given statement. Be prepared to explain your proposed exploration to the class.

Reasoning from patterns based on analysis of specific cases as you described in Problem 4 is called **inductive reasoning**. This type of reasoning is a valuable tool in making discoveries in mathematics, science, and everyday life. However, inductive reasoning must be used with caution.

5 The famous mathematician Leonard Euler (1707–1783) worked on a wide range of problems including questions of traversability of networks as you may have studied in Course 1. Like others of his time, he was interested in finding a formula to create prime numbers. An early attempt was:

If n is a positive integer, then
$n^2 - n + 41$ is a prime number.

a. Test this conjecture by examining some specific cases. Choose several positive values for n and see if the expression gives a prime number. Share the work with others.

b. Based on your calculations, does the conjecture seem correct? Can you conclude for sure that it is always true? Explain your reasoning.

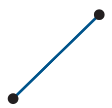
c. Test $n^2 - n + 41$ when $n = 41$. Is the result a prime number? Why or why not?

d. To prove an if-then statement is *not* true, you only have to find one counterexample. What is a counterexample to the statement, "if n is a positive integer, then $n^2 - n + 41$ is a prime number"?

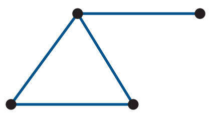
6 Now it is your turn to do some mathematical research. Recall that the *degree* of a vertex in a vertex-edge graph without loops is the number of edges touching the vertex. Use Parts a–c and inductive reasoning to develop a conjecture about the *sum* of the degrees of the vertices of a vertex-edge graph with no loops. Share the work with a partner.

a. For each of the graphs shown, determine the number of edges, the degree of each vertex, and the sum of the degrees of the vertices. Organize your results in a table. Leave room to extend your table vertically.

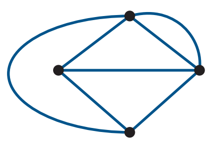
Graph I



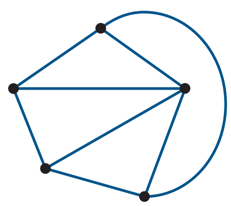
Graph II



Graph III



Graph IV



b. Draw four additional vertex-edge graphs without loops. Find the number of edges and the sum of the degrees of the vertices for each graph. Record your findings in your table.

c. If you have a vertex-edge graph with 10 edges and no loops, what do you think is the sum of the degrees of the vertices? Check your prediction with a drawing.

d. Write a conjecture relating the number of edges E and the sum S of the degrees of the vertices.

e. Write your conjecture in if-then form.

f. Can you be absolutely positive that your conjecture is true for all possible vertex-edge graphs with no loops? Explain.

Inductive reasoning may lead to an if-then statement that is plausible, or seems true. However, as you saw in Problem 5, the statement may not be true for all cases. **Deductive reasoning** involves reasoning *from* facts, definitions, and accepted properties *to* conclusions using principles of logic. Under correct deductive reasoning, the conclusions reached are certain, not just plausible.

7 In this problem, you will examine how deductive reasoning is used to *prove* the relationship you discovered in Problem 6. Compare your if-then statement in Part e of that problem with the following statement.

If G is a vertex-edge graph with E edges, none of which are loops, then the sum S of the degrees of the vertices is equal to 2E.

Here is an argument that is claimed to be a proof of this conjecture. Study it carefully.

If G is a vertex-edge graph with E edges, none of which are loops, then each of the E edges joins two vertices.

If each of the E edges joins two vertices, then each of the E edges contributes 2 to the sum S of the degrees of the vertices.

If each of the E edges contributes 2 to the sum of the degrees of the vertices S, then $S = 2E$.

Therefore, if G is a vertex-edge graph with E edges, none of which are loops, then the sum S of the degrees of the vertices is $2E$.

a. Does this argument convince you that the conjecture is correct? Why or why not?

b. Why do you think the argument started with, "If G is a vertex-edge graph with E edges, none of which are loops"?

c. Is the argument based on information that is known to be correct? Look critically at each of the first three if-then statements. Are they correct? Explain why or why not.

d. The argument is based on valid reasoning with a *chain of if-then statements*, given symbolically at the right. In the first implication $p \Rightarrow q$,

p is the statement, "G is a vertex-edge graph with E edges, none of which are loops,"

q is the statement, "each of the E edges joins two vertices."

Chaining Implications

$$\begin{array}{c} p \Rightarrow q \\ q \Rightarrow r \\ \underline{r \Rightarrow s} \\ ? \Rightarrow ? \end{array}$$

Analyze the form of the argument and then complete this symbolic representation. Identify the statement represented by each letter in the symbolic chain.

Summarize the Mathematics

Inductive reasoning and deductive reasoning are each important; they are complementary aspects of mathematical reasoning. Inductive reasoning often leads to conjectures of new relationships or properties that can be proven using deductive reasoning. Consider this conjecture.

The sum of any two consecutive odd numbers is divisible by 4.

a How could you arrive at this conjecture by using inductive reasoning?

b Write this conjecture in if-then form.

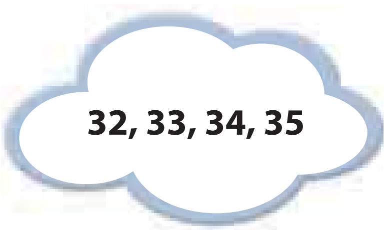
- **i.** What is the hypothesis of your statement?
- **ii.** What is the conclusion?

c How could you use deductive reasoning to prove this conjecture?

Be prepared to share your ideas and reasoning strategies with the entire class.

Check Your Understanding

Make a conjecture about what happens when you choose any four consecutive whole numbers, add the middle two, and then subtract the smallest of the four from that sum.



a. Describe the procedure you used to create your conjecture.

b. Write your conjecture in if-then form.

c. If n represents the smallest of four consecutive whole numbers, how would you represent each of the next three numbers?

d. Use your representations in Part c to write an argument that proves your conjecture is always true.

On Your Own

Applications

These tasks provide opportunities for you to use and strengthen your understanding of the ideas you have learned in the lesson.

1 Carefully study the following plot from a detective story.

Detectives Walker and Stanwick were called to the Engineering division of the Centipore medical building where it was discovered that someone had tampered with a sensitive computer disk. The security chief, Freedman, said he kept Engineering a restricted area. DiCampli, the vice-president of engineering, said that the disk was fine when used last Wednesday, the 11th, but was found to be altered last Friday, the 13th. Miller, a senior engineer, discovered the problem and immediately reported it to DiCampli who confirmed it and reported it to Parke, the executive vice-president of operations who, in turn, contacted the detectives.

Upon further investigation, the detectives learned that the door to the room housing the computer disk was unforced and thus must have been accessed by either DiCampli, Miller, Parke, or one of two other senior engineers, Donlan and Delaney. However, Donlan was new to the company and for now could only access the room with verification by Freedman, who was out with the flu for the past week. The company was able to verify use on the 11th and use on the 13th, when Miller discovered the problem. The disk was not used on the terminal between those times. Using the disk on another computer required a decryption code which was changed on the 9th and automatically issued to senior engineers. The code was issued to other higher-ups through Freedman.

Before arriving at Centipore, the detectives were given a tip that one of Centipore's competitors bribed one of the Centipore employees. With this information and the evidence they discovered at the crime scene, the detectives were sure they could figure out who the culprit was.

Adapted from: Smith, Stan. (2000). *Five-Minute Crimebusters: Clever Mini-Mysteries.* New York: Sterling Publishing Company, Inc. pp. 51–53.

a. Identify the prime suspect in this case. Explain the evidence you could use to charge that person and write a convincing argument to do so.

b. Write a series of arguments that could be used to exonerate or clear the other possible suspects of blame.

c. In what ways is the reasoning used to identify the prime suspect similar to the reasoning used to solve the Sudoku puzzle on page 2?

2 The following two-person game is played on a rectangular board, like a large index card. To play, players take turns placing pennies on the board. The coins may touch but cannot overlap or extend beyond the game board. A player cannot move the position of an already placed coin. The player who plays the last coin is the winner. The diagrams below show the first play of each of two players.

1st Player **2nd Player**

a. Play the game a few times with a partner, noting how the shape of the game board influences your play.

b. Use the symmetry of the game board to devise a strategy that will always result in a win for the *1st player* when he or she places the first coin in position A_1 as above.

c. On a copy of the second diagram at the right above, show the next play of a 1st player who is using your winning strategy. Then show the next two possible plays of each player.

d. Write a description of your winning strategy for the 1st player. Provide an argument for why that strategy will guarantee a win for the 1st player.

e. What other game board shapes can you make so that the strategy will work?

3 *Consecutive numbers* are adjacent integers on a number line, such as 5 and 6. Nathan, Trina, Kasib, and Ivana were trying to prove the following statement.

The sum of any two consecutive numbers is always an odd number.

Study each of the arguments below.

Nathan: If the first number is even, then the second number must be odd. This combination will always add up to an odd number.

Kasib: Two consecutive numbers are of the form n and $n + 1$.
$n + (n + 1) = 2n + 1$ which is, by definition, the form of an odd number.

Trina: No matter what two consecutive numbers you take, their sum is always odd as shown below.

$5 + 6 = 11$
$22 + 23 = 45$
$140 + 141 = 281$

Ivana: For any two consecutive numbers, one will be even and the other will be odd. In Part c of the Check Your Understanding task on page 9, I gave an argument justifying that the sum of an even number and an odd number is always an odd number. So, the sum of any two consecutive numbers is always an odd number.

a. Which proof is the closest to the argument you would give to prove that the sum of any two consecutive numbers is always an odd number?

b. Which arguments are *not* correct proofs of the statement? Explain your reasoning.

c. Give a "visual proof" of the statement using arrays of counters.

d. How would you prove or disprove the assertion, "The sum of three consecutive numbers is always an odd number"?

4 Examine each of the arguments below. Assuming the if-then statement is true, state whether the argument is correct or incorrect. Give a reason for your answers.

a. If the price of gas rises, then demand for gas falls. Demand for gas has risen. Therefore, the price of gas has risen.

b. If the price of gas rises, then demand for gas falls. The price of gas has risen. Therefore, the demand for gas has fallen.

c. If the price of gas rises, then demand for gas falls. The price of gas has fallen. Therefore, the demand for gas has fallen.

5 Suppose it is true that "all members of the senior class are at least 5 feet 2 inches tall." What, if anything, can you conclude with certainty about each of the following students?

a. Darlene, who is a member of the senior class

b. Trevor, who is 5 feet 10 inches tall

c. Anessa, who is 5 feet tall

d. Ashley, who is not a member of the senior class

6 Suppose it is true that "all sophomores at Calvin High School enroll in physical education."

a. Write this statement in if-then form. What is the hypothesis? The conclusion?

b. If Tadi is a sophomore at Calvin, what can you conclude?

c. If Rosa is enrolled in a physical education class at Calvin, what can you conclude? Explain your reasoning.

7 In 1742, number theorist Christian Goldbach (1690–1764) wrote a letter to mathematician Leonard Euler in which he proposed a conjecture that people are still trying to prove or disprove. Goldbach's Conjecture states:

Every even number greater than or equal to 4 can be expressed as the sum of two prime numbers.

a. Verify Goldbach's Conjecture is true for 12. For 28.

b. Write Goldbach's Conjecture in if-then form.

c. Write the converse of Goldbach's Conjecture. Prove that the converse is not true.

8 Examine the following if-then statements about properties of numbers.

I. If a, b, and c are consecutive positive numbers, then $a + b + c$ is divisible by 3.

II. If a, b, and c are consecutive positive numbers, then $a + b + c$ is divisible by 6.

III. If x is a real number, then $-x < x$.

IV. If x is a nonzero real number, then $x > \frac{1}{x}$.

V. If x is the degree measure of the smallest angle of a triangle, then $\cos x > 0$.

a. Use inductive reasoning to help you decide which statements might be correct and which are incorrect. For each statement that is incorrect, give a counterexample.

b. For each correct statement, use deductive reasoning to write a proof that could convince a skeptic that it is true.

9 Tonja made the following conjecture about consecutive whole numbers.

For any four consecutive whole numbers, the product of the middle two numbers is always two more than the product of the first and last numbers.

a. Test Tonja's conjecture for a set of four consecutive whole numbers.

b. Find a counterexample or give a deductive proof of Tonja's conjecture.

Connections

These tasks will help you connect the ideas of mathematical reasoning in this lesson with other mathematical topics and contexts that you know.

10 *Factorial* notation is a compact way of writing the product of consecutive positive whole numbers. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1$. 5! is read "5 factorial." In general, $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$.

a. Calculate 3!.

b. The names of three candidates for the same office are to be listed on a ballot. How many different orderings of the names are possible? Compare your answer to that found in Part a.

c. Provide an argument that $(n + 1)! = (n + 1) \times n!$.

d. Kenneth Ruthven of Cambridge University proposed the following conjectures about 100! to a class in Great Britain. For each conjecture, write an argument that proves the conjecture or explain why it is not true.

I. 100! is an even number.

II. 100! is divisible by 101.

III. 100! is larger than $101! - 100!$.

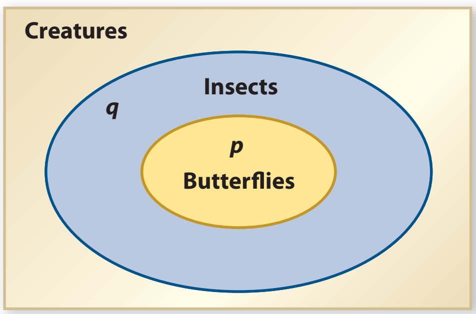
IV. 100! is larger than $50! \times 10^{50}$.

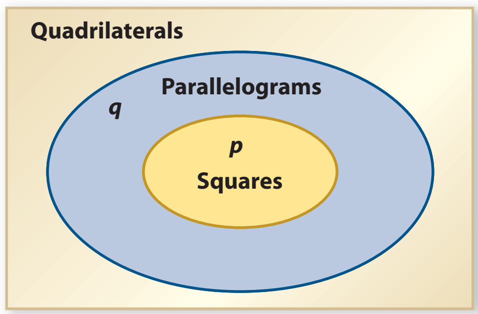
11 Problem 3 of Investigation 2 (page 12) illustrated a symbolic model for reasoning with an if-then statement. You can also represent if-then statements geometrically using *Venn diagrams*.

a. Examine these if-then statements and the corresponding Venn diagrams.

$p \Rightarrow q$
If a creature is a butterfly, then it is an insect.

$p \Rightarrow q$
If a quadrilateral is a square, then it is a parallelogram.





i. If you know that a monarch is a butterfly, what can you conclude?

ii. If you know that quadrilateral *WXYZ* is a square, what can you conclude?

b. Refer to Applications Task 6 (page 18). Represent the if-then statement you wrote in Part a with a Venn diagram. How can you use the Venn diagram to reason about Tadi's situation in Part b?

c. A common error in deductive reasoning is to assume that whenever an if-then statement $p \Rightarrow q$ is true, the *converse* statement $q \Rightarrow p$ is also true.

i. For each statement in Part a, write the converse and decide whether or not the converse statement is true. If not, give a counterexample.

ii. How do the Venn diagrams in Part a show that the converse of a statement is not always true? How is this analysis related to your reasoning about the case of Rosa in Applications Task 6 Part c (page 18)?

12 Suppose a statement $p \Rightarrow q$ and its converse $q \Rightarrow p$ are both true. Then the statement $p \Leftrightarrow q$ (read "p if and only if q") is true. In Course 1, you proved that both the Pythagorean Theorem and its converse were true. This fact can be stated in if-and-only-if form. *In* $\triangle ABC$, $\angle C$ *is a right angle if and only if* $a^2 + b^2 = c^2$.

a. Write the definition of a trapezoid from page 7 in if-and-only-if form.

b. Write the definition of a prime number from page 10 in if-and-only-if form.

c. Consider this statement about real numbers. If $a = b$, then $a + c = b + c$.

i. Write the converse of this statement.

ii. Is the converse always true? Explain your reasoning.

iii. Write an if-and-only-if statement summarizing this property of equality.

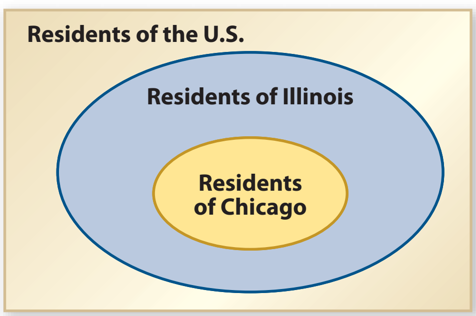
13 Consider the true statement, "If a person lives in Chicago, then the person lives in Illinois," and the corresponding Venn diagram below.

a. Which of the following statements are always true?

i. If a person does not live in Chicago, then the person does not live in Illinois.

ii. If a person does not live in Illinois, then the person does not live in Chicago.

b. How are your answers to Part a illustrated by the Venn diagram?



c. The if-then statement in Part ai is called the *inverse* of the original statement. In symbols, the inverse of $p \Rightarrow q$ is *not* $p \Rightarrow$ *not* q. Use a Venn diagram to explain why the inverse of a true if-then statement may not always be true.

d. The if-then statement in Part aii is called the *contrapositive* of the original statement. In symbols, the contrapositive of $p \Rightarrow q$ is *not* $q \Rightarrow$ *not* p. (An implication and its contrapositive are *logically equivalent statements*.) Use a Venn diagram to explain why the contrapositive of a true if-then statement is always true.

Reflections

These tasks provide opportunities for you to re-examine your thinking about ideas in the lesson.

14 Look back at Connor's strategy to guarantee that the second player can always win the nonagon game. (Investigation 1, page 5)

a. Will his strategy work if the game is played on the vertices of a regular pentagon? Explain your reasoning.

b. Will his strategy work if the game is played on the vertices of a regular octagon? Explain.

c. Describe as precisely as you can all regular polygons for which Connor's strategy will work.

15 Look back to page 6 at the arguments that Nesrin and Teresa provided to justify that the sum of two odd numbers is always an even number.

a. How can Nesrin's counter model help you to better understand Teresa's argument?

b. How could Nesrin's argument be revised to make it more general?

16 If-then statements are sometimes called *conditional statements*. Why does that term make sense?

17 In this age of the Internet and World Wide Web, advertising has become big business. Advertisers often use if-then statements to sell their products and services. The straightforward ad:

Use your money wisely—shop at FlorMart superstore.

is worded to suggest the implication:

If you shop at FlorMart, then you use your money wisely.

Often, with some added help from the advertiser, the statement is interpreted by consumers:

If you do not shop at FlorMart, then you do not use your money wisely.

a. Why might the wording of the second implication have a stronger psychological effect upon most shoppers than the first implication?

b. Are the two statements logically the same? Explain.

18 If you think about it, inductive reasoning is a common form of reasoning in the world around you. Give an example of how inductive reasoning might be used by the following people.

a. An automobile driver

b. A consumer

c. A medical researcher

19 Explain how inductive and deductive reasoning differ. In doing mathematics, how does one form of reasoning support the other?

Extensions

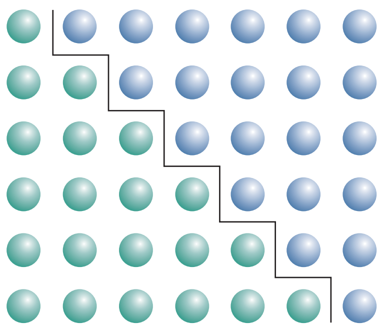
These tasks provide opportunities for you to explore further or more deeply the ideas you studied in this lesson.

20 Look back at the nonagon game in Investigation 1 (page 4). Another student, Sofia, claimed she found a strategy using symmetry that guaranteed that the *first player* could always win the game. What is wrong with her argument below?

As the first player, I'll remove the one penny at the top. Then in my mind, I divide the remaining 8 pennies by the line of symmetry determined by the removed penny. Now, whatever the other player does, I'll do the symmetric move. So, there is always a move for me to make. Therefore, I can never lose by having no coins to remove.

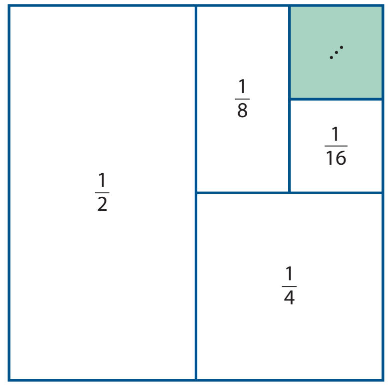
21 In his book, *Proofs without Words*, mathematician Roger Nelsen offers the following two visual "proofs." Although not proofs in the strictest sense, the diagrams he provides help you see why each particular mathematical statement is true.

a. How does the diagram below help you see that for any positive integer n, the sum of the integers from 1 to n is $\frac{1}{2}n(n + 1)$?



$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$

b. How does the diagram at the right help you see that the "infinite sum" of fractions of the form $\left(\frac{1}{2}\right)^n$, $n \geq 1$ is 1?

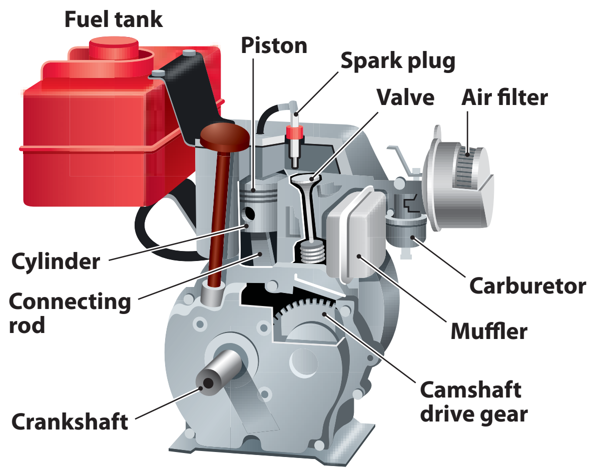


$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

22 Many people, such as auto technicians, make a living out of repairing things. Often, repairs can be done at home if you have the right tools and can reason deductively. The first step in making a repair is to identify the problem. Examine the troubleshooting chart below for diagnosing problems that often occur with small engines, such as the one on a lawn mower.

a. What are the first things you should check if the engine runs but the mower does not?

Four-cycle engine with horizontal crankshaft



Starter doesn't turn engine	Engine turns, will not start	Engine starts, then stalls	Runs unevenly, lacks power	Engine stalls at idle speed	Engine dies during use	Engine runs too fast	Engine runs, device doesn't	Engine vibrates excessively	Engine smokes profusely	Engine overheats	Engine backfires	Engine is extra-noisy	POSSIBLE CAUSE
●													Starter rope jammed or broken
●													Recoil starter spring weak or broken
●													Electric starter battery low, connections loose or corroded, or power cord damaged
		●	●	●	●				●				Clogged air filter
	●	●			●								Fuel tank empty; add fuel
	●	●	●	●	●								Contaminated fuel
		●	●		●								Fuel cap breather holes blocked
	●	●	●		●								Clogged fuel filter
	●	●			●								Obstructed or damaged fuel line
	●	●	●	●					●				Sticky choke; clean with solvent (you may have to disassemble the carburetor)
		●		●									Idle speed incorrect; see owner's manual to adjust
			●		●	●							Governor linkage out of adjustment
	●	●	●	●	●	●			●	●	●		Carburetor fouled or poorly adjusted
	●	●	●		●				●	●			On two-cycle engine, incorrect fuel mixture
	●		●	●							●		On two-cycle engine, faulty reed valve
	●	●	●	●	●								Fouled spark plug or damaged cable
					●					●		●	Insufficient oil in crankcase; add oil to oil tank
			●		●								Overheated; allow engine to cool, then restart
			●								●	●	Valves sticking, burned, or need adjusting
	●	●	●		●					●	●		Breaker points dirty or improperly set
	●	●	●	●	●				●				Low compression
		●	●	●	●					●		●	Clogged muffler or exhaust port
							●						Slipping belt
							●	●					Faulty clutch
										●			Cooling fins dirty; clean fins
	●	●	●	●	●			●			●		Flywheel key bent, broken, or worn
								●				●	Flywheel loose or fins damaged; tighten or replace
	●	●						●					Crankshaft bent; have engine replaced

b. What should you check if the engine runs too fast? If it backfires?

c. Some of the "possible causes" listed in the troubleshooting chart often suggest a next course of action. Explain how if-then reasoning is used in these cases.

23 Recall that the basic if-then reasoning pattern, Affirming the Hypothesis, can be represented as shown on the left below. In symbolic form, everything above the horizontal line is assumed to be correct or true. What is written below the line follows logically from the accepted information.

Affirming the Hypothesis		**Denying the Conclusion**
$p \Rightarrow q$	$\text{not } q \Rightarrow \text{not } p$	$p \Rightarrow q$
$\underline{p \quad\quad}$	$\underline{\text{not } q \quad\quad\quad}$	$\underline{\text{not } q \quad\quad\quad}$
q	$\text{not } p$	$\text{not } p$

a. Explain why the second reasoning pattern above is valid.

b. Use the result of Connections Task 13 Part d to explain why Denying the Conclusion, shown above on the right, is a valid reasoning pattern.

c. Use Denying the Conclusion to decide what can be concluded from the following two statements.

Known fact: If a triangle is an isosceles triangle, then it has two sides the same length.

Given: $\triangle PQR$ has no pair of sides the same length.

Conclusion: ?

24 *Number theory* is a branch of mathematics that has flourished since ancient times and continues to be an important field of mathematical activity, particularly in the applied area of *coding* (encrypting messages so only the intended recipients can read them). One of the first definitions appearing in the theory of numbers is a definition for *factor* or *divisor*. An integer b is a factor or divisor of an integer a provided there is an integer c such that $a = bc$.

a. One of the first theorems in number theory follows.

If a, b, and c are integers where a is a factor of b and a is a factor of c, then a is a factor of b + c.

Test this theorem for some specific cases to develop an understanding for what it says. Then write a deductive argument to prove that the theorem is always true.

b. Form a new if-then statement as follows. Use the hypothesis of the theorem in Part a, and replace the conclusion with "a is a factor of $bm + cn$ for all integers m and n."

i. Do you think this new if-then statement is always true? Explain your reasoning.

ii. If you think the statement is true, write a proof of it. If not, give a counterexample.

c. Prove or disprove this claim.

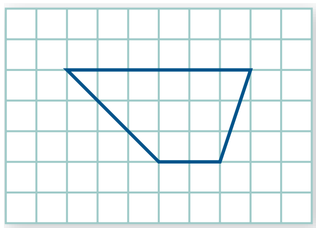
If a is a factor of b and b is a factor of c, then a is a factor of c.

Review

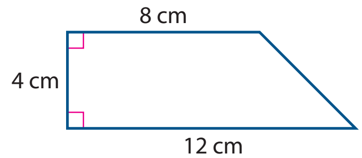
These tasks provide opportunities for you to review previously learned mathematics and to refine your skills in using that mathematics.

25 Find the area of each trapezoid.

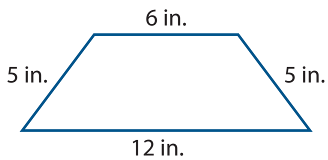
a.



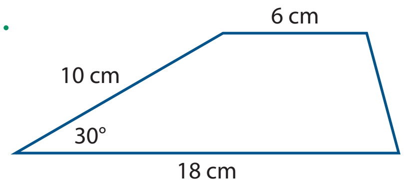
b.



c.



d.



26 Evaluate each expression for the values $x = \frac{5}{6}$, $y = -3$, and $z = 6$.

a. xz^2

b. x^{-1}

c. $|y - z| + y^2$

d. $x + \frac{4}{y}$

27 Determine if the pairs of lines are parallel, perpendicular, or neither.

a. The line containing $(8, 2)$ and $(-4, 6)$ and the line containing $(-2, 3)$ and $(1, 4)$

b. The lines with equations $y = 6x + 5$ and $y = -6x - 5$

c. The lines with equations $4x + 3y = 10$ and $3x - 4y = 10$

28 Rewrite each expression in simpler equivalent form.

a. $7x - 2x$

b. $3(2x + 5) - 7$

c. $(4a - 9) - 3(5a + 1)$

d. $\frac{15x}{3}$

e. $\frac{10x + 15}{5}$

f. $\frac{4t - 7}{4}$

29 Recall that the converse of an if-then statement reverses the order of the two parts of the statement. Consider this statement.

If a polygon is a regular polygon, then all its sides are the same length.

a. Is this a true statement?

b. Write the converse of this statement.

c. Is the converse a true statement?

30 Write each of these expressions in equivalent expanded form.

a. $5(2x + 7)$

b. $(x + 9)(x + 3)$

c. $(x - 4)(x + 3)$

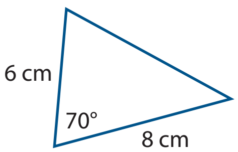
d. $(x + 4)(x - 4)$

e. $(x + 3)^2$

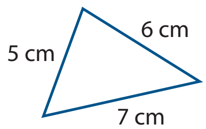
f. $(x - 3)^2$

31 Is the information in each diagram below enough to completely determine the size and shape of the triangle? Explain your reasoning in each case.

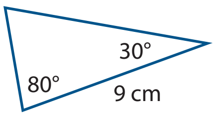
a.



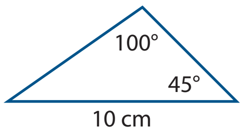
b.



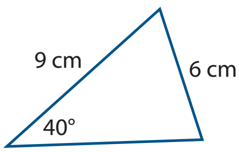
c.



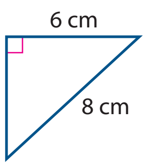
d.



e.



f.

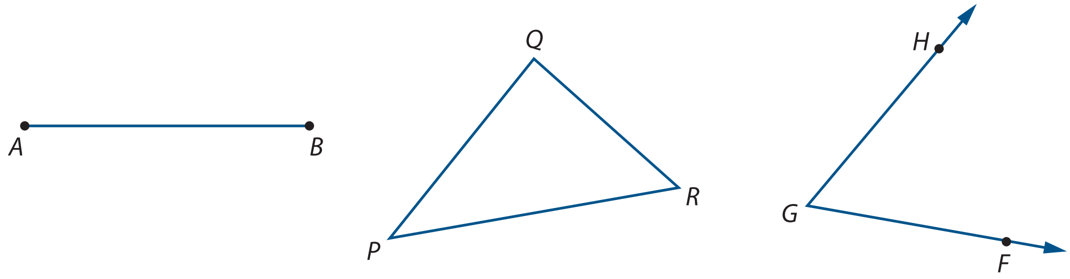


32 Using a compass and straightedge (no measuring), construct each of the following.

a. A segment congruent to $\overline{AB}$

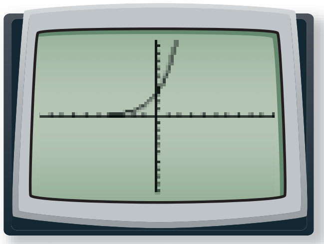
b. A triangle congruent to $\triangle PQR$

c. An angle congruent to $\angle FGH$

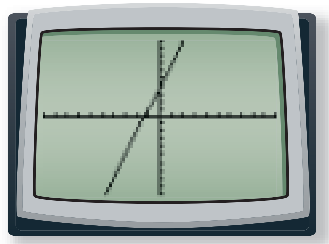


33 Without using your calculator, match each graph with the correct function rule. The scales on all axes are in increments of 1 unit. Check your work by graphing the functions using your calculator.

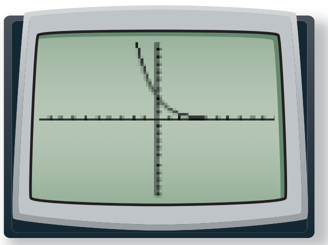
I



II



III



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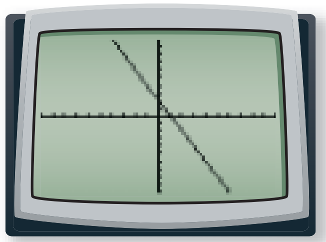

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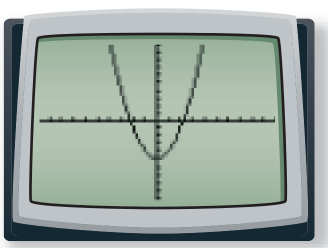
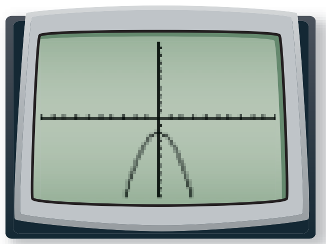


VI

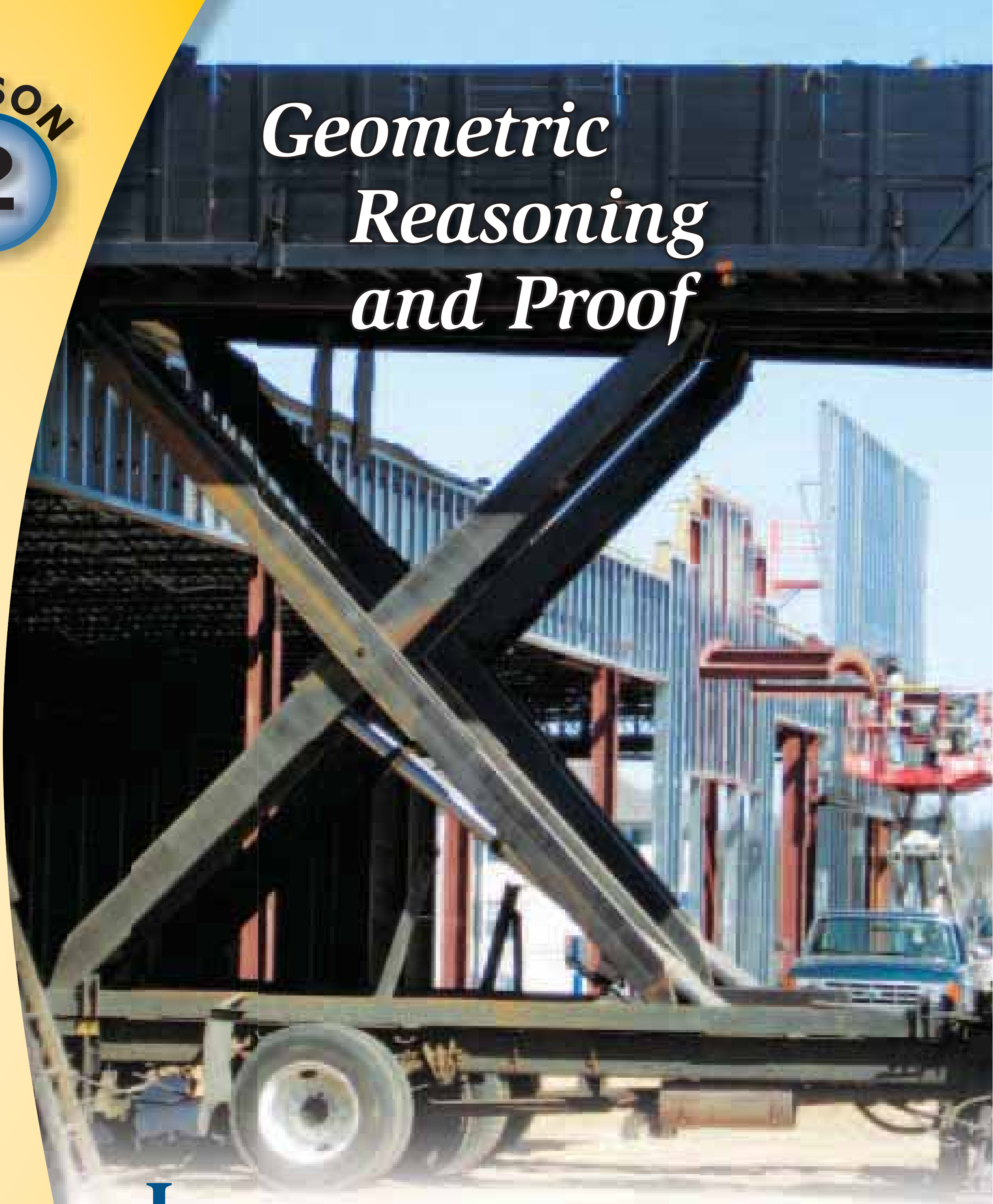


a. $y = 3x + 4$ **b.** $y = 3(2^x)$ **c.** $y = -x^2 - 2$

d. $y = -2x + 2$ **e.** $y = 3(0.5^x)$ **f.** $y = x^2 - 5$

LESSON 2

Geometric Reasoning and Proof



In your previous studies, you used informal geometric reasoning to help explain the design of structures and the functioning of mechanisms. You also explored how if-then reasoning could be used to derive geometric properties. For example, you saw that if you know the formula for the area of a rectangle, you can then derive formulas for the area of a square and of a right triangle. Knowing how to calculate the areas of these shapes enabled you to complete a proof of the Pythagorean Theorem.

The interplay among shapes, their properties, and their function is a recurring theme in geometry. The design of the lift-bed truck shown above is based on two of the most common figures in a plane, lines and angles.

Think About This Situation

Analyze the lift-bed truck mechanism shown on the previous page.

a. How do you think the mechanism works?

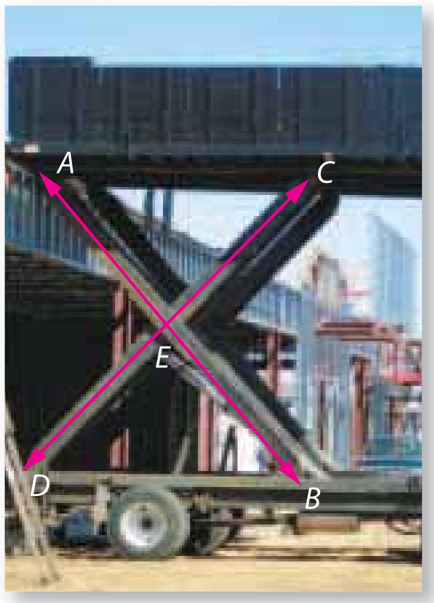
b. As the truck bed is raised or lowered, what elements change?

c. As the truck bed is raised or lowered, what segment lengths and what angle measures remain unchanged?

d. As the truck bed is raised or lowered, will it always remain parallel to the flat-bed frame of the truck? Explain your reasoning.

In this lesson, you will explore how from a few basic assumptions you can prove important properties of angles formed by intersecting lines and by two parallel lines intersected by a third line.

Investigation 1 Reasoning about Intersecting Lines and Angles



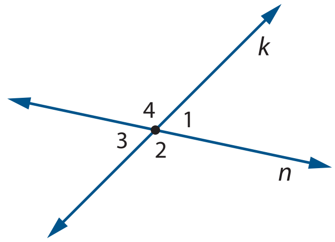
Skill in reasoning, like skill in sculpting, playing a musical instrument, or playing a sport, comes from practicing that skill and reflecting on the process. In this lesson, you will sharpen your reasoning skills in geometric settings. As you progress through the lesson, pay particular attention to the assumptions you make to support your reasoning, as well as to the validity of your reasoning.

When two lines intersect at a single point, special pairs of angles are formed. For example, a pair of adjacent angles formed by two intersecting lines like $\angle AEC$ and $\angle CEB$, shown at the right, are called a **linear pair** of angles. Pairs of angles like $\angle AEC$ and $\angle BED$ are called **vertical angles**. As you work on the problems in this investigation, make notes of answers to these questions:

How are linear pairs of angles related?

How are vertical angles related and why is that the case?

1 In the diagram at the right, lines k and n intersect at the point shown, forming angles numbered 1, 2, 3, 4.



a. If $m\angle 1 = 72°$, what can you say about $m\angle 2$? About $m\angle 3$? About $m\angle 4$? What assumptions are you using to obtain your answers?

b. If $m\angle 2 = 130°$, what can you say about $m\angle 1$? About $m\angle 3$? About $m\angle 4$?

c. In general, what relationships between pairs of angles do you think are true? Make a list of them.

d. Will the general relationships you listed for Part c hold for any pair of intersecting lines? Test your conjectures using specific examples.

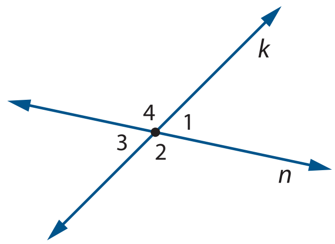
e. Write an if-then statement about linear pairs of angles that you think is *always* correct. You may want to begin as follows. If two angles are a linear pair, then … .

f. Write an if-then statement about vertical angles that you think is *always* correct. You may want to begin as follows. If two lines intersect, then … .

In the remainder of this lesson, you will continue to use inductive reasoning to discover possible relations among lines and angles, but you will also use deductive reasoning to prove your conjectures are always true. To reason deductively, you must first have some basic facts from which to reason. In mathematics, statements of basic facts that are accepted as true without proof are called **postulates** (or *axioms*). These assumed facts will be helpful in supporting your reasoning in the remainder of this unit and in future units. Begin by assuming the following postulate concerning linear pairs of angles.

Linear Pair Postulate If two angles are a linear pair, then the sum of their measures is 180°.

2 Study the attempt at the right by one group of students at Washington High School to prove the conjecture they made in Part f of Problem 1. Based on the labeling of the diagram, they set out to prove the following.



If lines n and k intersect at the point shown, then $m\angle 1 = m\angle 3$.

They reasoned as follows.

(1) Since lines n and k intersect, $\angle 1$ and $\angle 2$ are a linear pair. So, $m\angle 1 + m\angle 2 = 180°$.

(2) Since lines n and k intersect, $\angle 2$ and $\angle 3$ are a linear pair. So, $m\angle 2 + m\angle 3 = 180°$.

(3) If $m\angle 1 + m\angle 2 = 180°$ and $m\angle 2 + m\angle 3 = 180°$, then $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$.

(4) If $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$, then $m\angle 1 = m\angle 3$.

a. Explain why each of the statements in the students' reasoning is or is not correct.

b. Now write an argument to show the following: If lines n and k intersect at the point shown, then $m\angle 2 = m\angle 4$. Give reasons justifying each of your statements.

In mathematics, a statement that has been proved using deductive reasoning from definitions, accepted facts, and relations is called a **theorem**. The statement proved in Problem 2 is sometimes referred to as the **Vertical Angles Theorem**, *vertical angles have equal measure*.

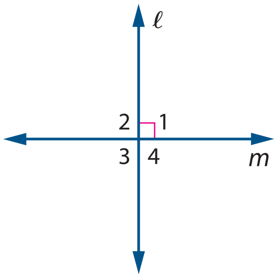
In Course 1, you proved the Pythagorean Theorem. (Not all theorems are given names.) After a theorem has been proved, it may be used to prove other conjectures. As your geometric work in Course 3 progresses, you will want to know which theorems have been proved. Thus, you should prepare a geometry toolkit by listing assumptions, such as the Linear Pair Postulate, and proven theorems. Add each new theorem to your toolkit as it is proved.

3 Recall that two intersecting lines (line segments or rays) are **perpendicular** ($\perp$) if and only if they form a right angle.

a. Rewrite this definition as two if-then statements.

b. Claim: *Two perpendicular lines form four right angles.* Is this claim true or false? Explain your reasoning.

c. Study the following strategy that Juanita used to prove the claim in Part b.



- First, she drew and labeled the diagram at the right.
- Then she developed a *plan for proof* based on her diagram.

I know that if $\ell \perp m$, they form a right angle, say $\angle 1$. A right angle has measure 90°. Use the fact that $\angle 1$ and $\angle 3$ are vertical angles to show $m\angle 3 = 90°$. Use the fact that $\angle 1$ and $\angle 2$ are a linear pair to show $m\angle 2 = 90°$. Then use the fact that $\angle 2$ and $\angle 4$ are vertical angles to show $m\angle 4 = 90°$.

- She then wrote her proof in a *two-column statement-reason form*.

Statements	Reasons
1. $\ell \perp m$	1. Given
2. ℓ and m form a right angle. Call it $\angle 1$.	2. Definition of perpendicular lines
3. $m\angle 1 = 90°$	3. Definition of right angle
4. $\angle 1$ and $\angle 3$ are vertical angles.	4. Definition of vertical angles
5. $m\angle 3 = m\angle 1 = 90°$	5. Vertical Angles Theorem
6. $\angle 1$ and $\angle 2$ are a linear pair.	6. Definition of linear pair
7. $m\angle 1 + m\angle 2 = 180°$	7. ________________
8. $m\angle 2 = 180° - m\angle 1 = 90°$	8. ________________
9. $\angle 2$ and $\angle 4$ are vertical angles.	9. ________________
10. $m\angle 2 = m\angle 4$	10. ________________
11. $m\angle 4 = 90°$	11. ________________
12. $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$ are right angles.	12. ________________

i. How does the diagram that Juanita drew show the information given in the claim?

ii. Why might it be helpful to develop a plan for a proof before starting to write the proof?

iii. Check the correctness of Juanita's reasoning and supply reasons for each of statements 7–12.

iv. Describe a *plan for proof* of the above claim that does *not* involve use of the Vertical Angles Theorem.

4 The design of buildings often involves perpendicular lines. In preparing a plan for a building like that below, an architect needs to draw lines perpendicular to given lines.

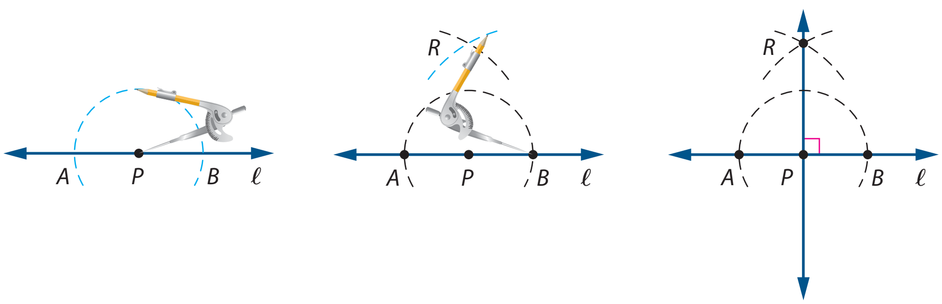
Marx and O'Neill Architecture

a. Draw a line ℓ on a sheet of paper. Describe how you would draw a line perpendicular to line ℓ through a point P in each case below.

i. P is a point on line ℓ.

ii. P is a point not on line ℓ.

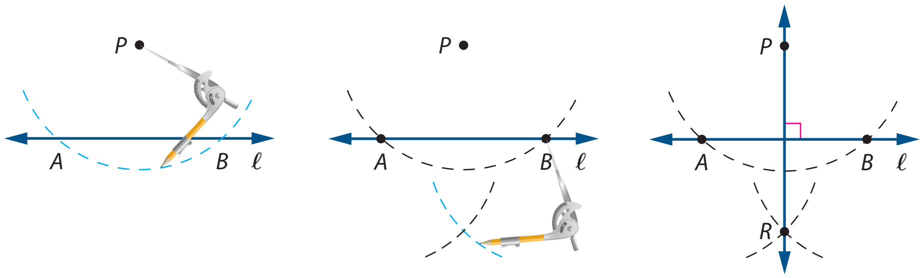
b. Study the diagrams below which show a method for constructing a line perpendicular to a given line ℓ through a given point P on line ℓ. In the second step, the same compass opening is used to create both arcs.



i. On a separate sheet of paper, draw a line ℓ and mark a point P on ℓ. Use a compass and straightedge to construct a line perpendicular to line ℓ at point P.

ii. Write an argument justifying that $\overleftrightarrow{PR} \perp \ell$ in the diagram above on the right. Start by showing that $\triangle APR \cong \triangle BPR$.

c. The following diagrams show a method for constructing a line perpendicular to a given line ℓ through a point P *not* on line ℓ. In the second step, the same compass opening is used to create both arcs.

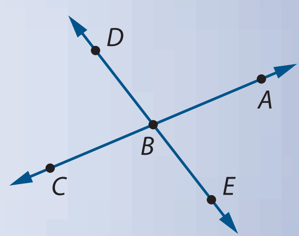


i. On a separate sheet of paper, draw a line ℓ and a point P not on ℓ. Construct a line perpendicular to line ℓ through point P.

ii. Write an argument justifying that $\overleftrightarrow{PR} \perp \ell$ in the diagram above on the right.

Summarize the Mathematics

In this investigation, you used deductive reasoning to establish relationships between pairs of angles formed by two intersecting lines. In the diagram at the right, suppose the lines intersect so that m∠*DBA* = m∠*CBD*.



a What can you conclude about these two angles? Prepare an argument to prove your conjecture.

b What can you conclude about the other angles in the diagram? Write a proof of your conclusion.

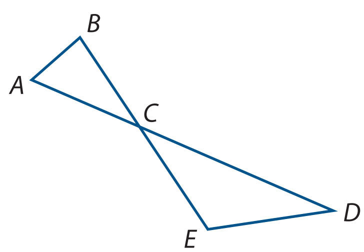
c What mathematical facts did you use to help prove your statements in Parts a and b? Were these facts definitions, postulates, or theorems?

d Describe the relationship between $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$.

Be prepared to share your conjectures and explain your proofs.

Check Your Understanding

In the diagram at the right, $\overline{AD}$ and $\overline{BE}$ intersect at point C and $m\angle ECD = m\angle D$.



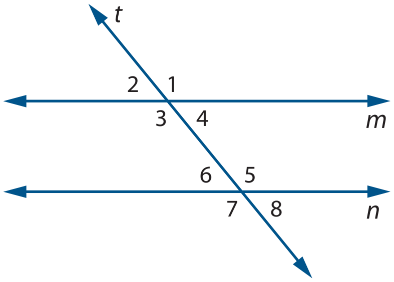
a. Is $m\angle ACB = m\angle D$? If so, prove it. If not, give a counterexample.

b. Is $m\angle A = m\angle D$? If so, prove it. If not, give a counterexample.

Investigation 2 Reasoning about Parallel Lines and Angles

When a line intersects another line, four angles are formed. Some of the pairs of angles have equal measures, and some pairs are **supplementary angles**—they have measures that add to 180°. When a line intersects *two* lines, many more relationships are possible. Perhaps the most interesting case is when a line intersects two parallel lines, as with the various pairs of support beams on the faces of the John Hancock Center in Chicago, shown below.

Lines in a plane that do not intersect are called **parallel lines**. In the diagram below, line m is parallel to line n (written $m \parallel n$). Line t, which intersects the two lines, is called a **transversal**.



As you work on the problems of this investigation, look for answers to the following questions:

If two parallel lines are intersected by a transversal, what relations exist among the measures of the angles formed?

What relations among the angles formed when two lines are cut by a transversal allow you to conclude that the lines are parallel?

1 In the preceding diagram, the angles at each point of intersection are numbered so that they can be easily identified.

a. What pairs of angles, if any, appear to be equal in measure?

b. What angle pairs appear to be supplementary? (Supplementary angles need not be a linear pair.)

c. Draw another pair of parallel lines and a transversal with a slope different from the one above. Number the angles as in the figure above.

i. Do the same pairs of numbered angles appear equal in measure?

ii. Do the same pairs of numbered angles appear to be supplementary?

Angles that are in the same relative position with respect to each parallel line and the transversal are called **corresponding angles**. In the diagram on the previous page, angles 1 and 5 are corresponding angles; similarly, angles 3 and 7 are corresponding angles.

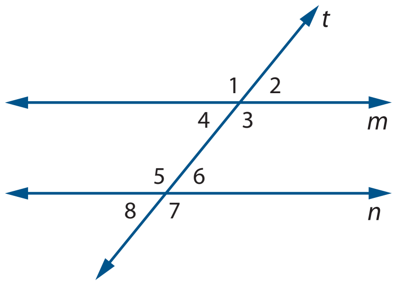
2 Examine the diagram you drew for Part c of Problem 1.

a. Name two pairs of corresponding angles, other than angles 1 and 5 or angles 3 and 7. Were those corresponding angles among the pairs of angles that you thought had equal measure?

b. Suppose $m\angle 1 = 123°$. Find the measures of as many other angles as you can in your diagram.

3 Descriptive names are also given to other pairs of angles formed by a transversal and two parallel lines. In the diagram below, $m \parallel n$ and t is a transversal intersecting m and n.

a. For each pair of angles named below, describe how the pair can be identified in a diagram. Then give one more example of such a pair.



i. **Interior angles on the same side of the transversal:** $\angle 4$ and $\angle 5$

ii. **Exterior angles on the same side of the transversal:** $\angle 2$ and $\angle 7$

iii. **Alternate interior angles:** $\angle 4$ and $\angle 6$

iv. **Alternate exterior angles:** $\angle 1$ and $\angle 7$

b. Identify a relationship that seems to exist for each type of angle pair named in Part a. Write your observations in if-then form, beginning each statement as follows. If two parallel lines are cut by a transversal, then … .

For Problems 4 and 5, assume the following statement as a known fact.

Corresponding Angles Assumption If two parallel lines are cut by a transversal, then corresponding angles have equal measure.

4 In completing Problem 3 Part b, one group of students at Brookwood High School made the following claim.

If two parallel lines are cut by a transversal, then interior angles on the same side of the transversal are supplementary.

a. Describe a plan for how you would prove this claim using the diagram in Problem 3 and the Corresponding Angles Assumption. Compare your plan for proof with that of others. Correct any errors in reasoning.

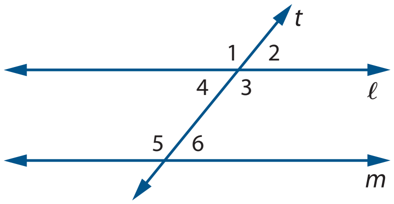
b. The start of a proof given by the group of Brookwood students is on the next page.

i. Supply a reason for each statement.

ii. Continue the two-column statement-reason proof to show that $\angle 3$ and $\angle 6$ are supplementary.

Given: $\ell \parallel m$; t is a transversal cutting ℓ and m

Prove: $\angle 4$ and $\angle 5$ are supplementary. $\angle 3$ and $\angle 6$ are supplementary.



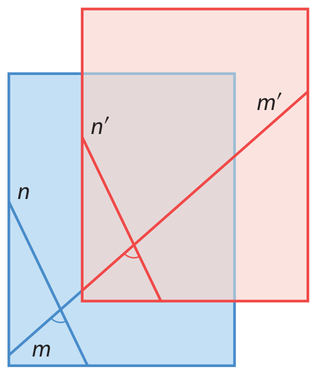
Statements	Reasons
1. $\ell \parallel m$; t is a transversal cutting ℓ and m	1.
2. $m\angle 4 + m\angle 1 = 180°$	2.
3. $m\angle 1 = m\angle 5$	3.
4. $m\angle 4 + m\angle 5 = 180°$	4.
5. $\angle 4$ and $\angle 5$ are supplementary.	5.
⋮	⋮

5 Describe plans for how you would prove that each of your three remaining conjectures in Part b of Problem 3 is correct. Share the task with others. Then discuss each other's plans for proof. Correct any errors in reasoning.

Using the Corresponding Angles Assumption, you can conclude that if two parallel lines are cut by a transversal, then certain relations among pairs of angles will always be true. In the next problem, you will consider the converse situation.

What relations among the angles formed when two lines are cut by a transversal allow you to conclude that the lines are parallel?

6 Conduct the following experiment.



a. Draw and label two intersecting lines on a sheet of paper and mark an angle as shown.

b. Trace the lines and the angle marking on a second sheet of paper. Label the corresponding lines m' and n'. Slide the top copy so that line m' is a continuation of line m.

c. Why are the two marked angles congruent?

d. How is the pair of marked angles related to the two lines n and n' and transversal m? How do lines n and n' appear to be related? Check if those relationships hold when you slide the top copy to other positions, keeping line m' as a continuation of line m.

e. Write a conjecture in if-then form that generalizes the observations you made in the experiment.

In order to reason deductively about parallel lines and figures formed by parallel lines, you need to begin with some information about the conditions under which two lines are parallel. The conjecture you made in Problem 6 Part e could be stated this way.

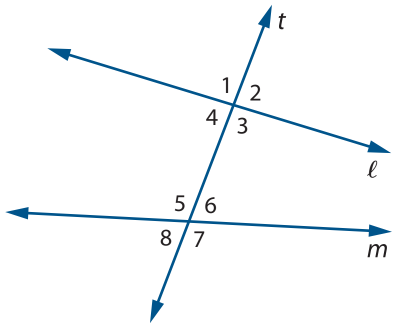
If two lines are cut by a transversal so that corresponding angles have equal measure, then the lines are parallel.

This statement is the *converse* of the Corresponding Angles Assumption. There, you assumed that if two parallel lines are cut by a transversal, then corresponding angles have equal measure.

For the remainder of this unit and in future units, you can assume that both the Corresponding Angles Assumption and its converse are true. These two statements are combined as a single if-and-only-if statement called the *Parallel Lines Postulate*.

Parallel Lines Postulate In a plane, two lines cut by a transversal are parallel if and only if corresponding angles have equal measure.

7 It is reasonable to ask if there are other relations between two angles formed by a line intersecting two other lines that would allow you to conclude that the two lines are parallel. Consider the diagram below.



a. What condition on a pair of alternate interior angles would guarantee that line ℓ is parallel to line m? Write your conjecture in if-then form.

b. What condition on a pair of interior angles on the same side of the transversal t would guarantee that line ℓ is parallel to line m? Write your conjecture in if-then form.

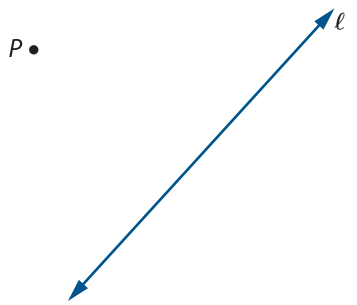
c. What condition on a pair of exterior angles would guarantee that line ℓ is parallel to line m? Write your conjecture in if-then form.

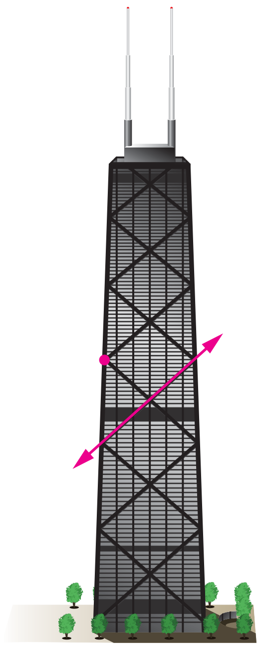
d. Working with a classmate, write a proof for one of the statements in Parts a–c.

e. Be prepared to share and discuss the reasoning in your proof with the entire class. Correct any reasoning errors found.

8 In designing buildings such as the John Hancock Center, architects need methods for constructing parallel lines. In Investigation 1 (page 34), you examined how to use a compass and straightedge to construct a line perpendicular to a given line through a point not on the line.

a. On a copy of the diagram below, show how you could use the constructions of perpendiculars (pages 33–34) to construct a line m through point P that is parallel to ℓ.





b. Write an argument that justifies that $m \parallel \ell$.

Summarize the Mathematics

In this investigation, you reasoned both inductively and deductively about angles formed by parallel lines and a transversal.

a What statements did you accept to be true without proof?

b What theorems and their converses were you able to prove about parallel lines and the angles they form with a transversal?

c Restate each theorem and its converse in Part b as a single if-and-only-if statement similar to the statement of the Parallel Lines Postulate.

Be prepared to compare your responses with those of others.

Check Your Understanding

Each of the following statements expresses a relationship between perpendicular and parallel lines. For each statement, draw and label a diagram. Then write an argument proving that the statement is true.

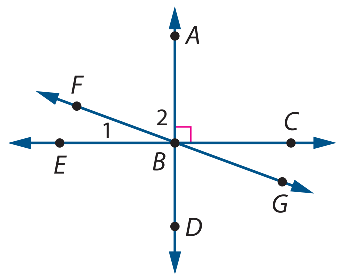
a. In a plane, if two lines are perpendicular to the same line, then they are parallel.

b. If a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the other.

On Your Own

Applications

1 In the diagram at the right, the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{EC}$, and $\overleftrightarrow{FG}$ intersect at point B. $\overleftrightarrow{AD} \perp \overleftrightarrow{EC}$.



a. What is m$\angle DBC$?

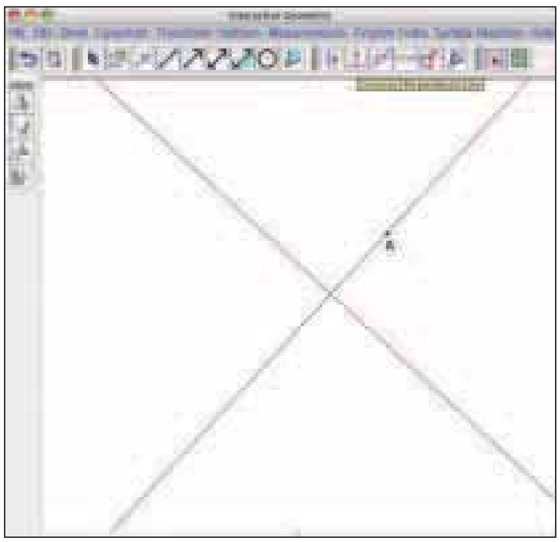
b. Suppose m$\angle 1 = 27°$. Find the measure of each angle.

- **i.** $\angle 2$
- **ii.** $\angle FBD$
- **iii.** $\angle EBG$

c. Suppose m$\angle 2 = 51°$. Find m$\angle ABG$ and m$\angle 1$.

d. How would your answers to Part c change if m$\angle 2 = p$, where $0° < p < 90°$?

2 Computer-aided design (CAD) programs and interactive geometry software include tools for constructing perpendicular lines and parallel lines.



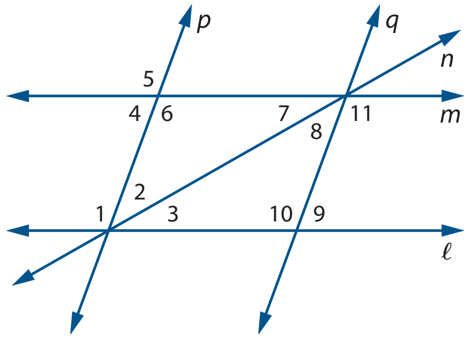
a. Use one of those tools to draw a line. Then construct a line through a point A perpendicular to the drawn line in each case below.

- **i.** A is a point on the line.
- **ii.** A is a point *not* on the line.

b. How do you think your software determines the perpendicular line in each case?

c. Use one of those tools to construct a line parallel to a drawn line through a point not on the line. How do you think the software determines the parallel line?

3 Use the diagram below with separate assumptions for Part a and Part b.



a. Assume $\ell \parallel m$, $p \parallel q$, $m\angle 2 = 40°$, and $m\angle 3 = 35°$.

i. Find $m\angle 8$. **ii.** Find $m\angle 10$.

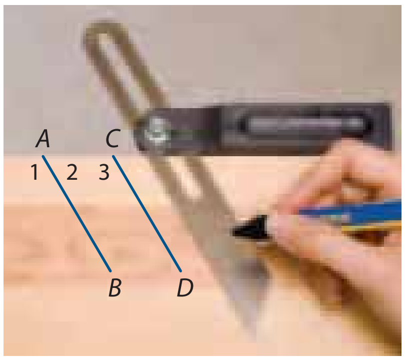
iii. Find $m\angle 4$. **iv.** Find $m\angle 7$.

b. Do not assume any of the given lines are parallel. For each of the given conditions, which lines, if any, can you conclude are parallel?

i. $m\angle 2 = m\angle 8$ **ii.** $m\angle 6 = m\angle 1$

iii. $m\angle 1 = m\angle 10$ **iv.** $m\angle 2 = m\angle 7$

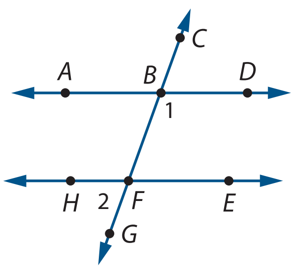
4 The photo below shows a carpenter's bevel, which is used to draw parallel lines.



a. What part of the bevel guarantees that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$?

b. How are $\angle 2$ and $\angle 3$ related? How do you know?

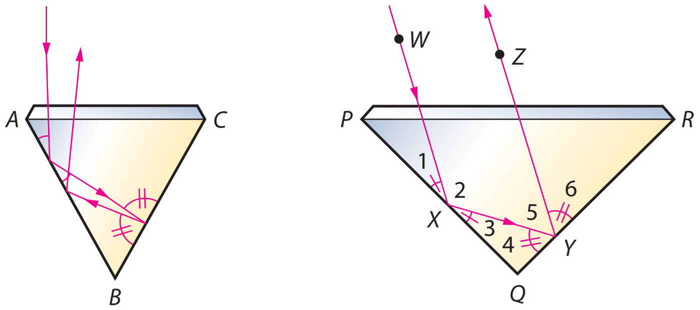
5 In the diagram below, $\overleftrightarrow{AD}$ and $\overleftrightarrow{HE}$ are cut by transversal $\overleftrightarrow{GC}$. $\angle 1$ and $\angle 2$ are supplementary.



Can you conclude $\overleftrightarrow{AD} \parallel \overleftrightarrow{HE}$? If so, prove it. If not, find a counterexample.

6 The sparkle of a diamond results from light being reflected from facet to facet of the jewel and then directly to the eye of the observer. When a light ray strikes a smooth surface, such as a facet of a jewel, the angle at which the ray strikes the surface is congruent to the angle at which the ray leaves the surface. A diamond can be cut in such a way that a light ray entering the top will be parallel to the same ray as it exits the top.

Examine the cross sections of the two diamonds shown below. The diamond at the left has been cut too deeply. The entering and exiting light rays are not parallel. The diamond on the right appears to be cut correctly. The entering and exiting light rays are parallel.



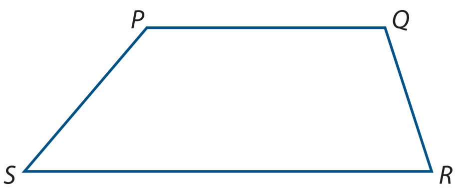
a. Use deductive reasoning to determine the measure of $\angle Q$ at the base of the diamond to ensure that the entering and exiting light rays will be parallel.

b. Explain how you could use inductive reasoning to conjecture what the measure of $\angle Q$ should be to ensure that the entering and exiting light rays will be parallel.

c. What are the advantages of deductive reasoning in this case?

d. As you have seen before, when modeling a situation it often helps to make simplifying assumptions. Consult with a science teacher about reflection and refraction of light and the modeling of the diamond cut. What simplifying assumptions were made in this situation?

7 Figure *PQRS* is a trapezoid with $\overline{PQ} \parallel \overline{SR}$.

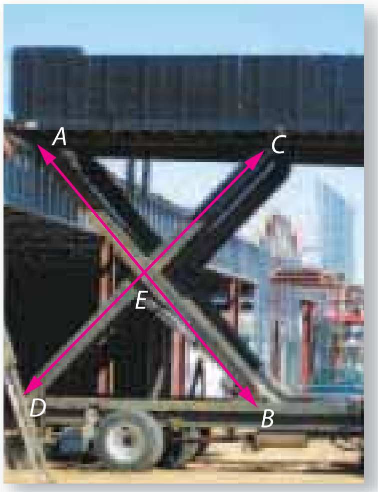


a. What can you conclude about $\angle S$ and $\angle P$? About $\angle Q$ and $\angle R$?

b. Write an if-then statement summarizing one of your observations in Part a. Prove your statement is correct.

c. Suppose $m\angle S = m\angle R$. What can you conclude about $\angle P$ and $\angle Q$? Write an argument proving your claim.

8 Re-examine the mechanism of the lift-bed truck shown here. Assume the frame of the truck is parallel to the ground, and E is the midpoint of $\overline{AB}$ and $\overline{DC}$. You may want to make a simple model by attaching two linkage strips of the same length at their midpoints.



a. How is $\angle AEC$ related to $\angle BED$? Why?

b. How is $\triangle AEC$ related to $\triangle BED$? Why?

c. How is $\angle BAC$ related to $\angle ABD$? Why?

d. As a hydraulic ram pushes point B in the direction of point D, what happens to the relations in Parts a–c? Explain your reasoning.

e. Explain as precisely as you can why the truck bed $\overline{AC}$ remains parallel to the truck frame $\overline{BD}$ as the hydraulic ram is extended or contracted.

Connections

9 When two lines intersect, two pairs of vertical angles are formed.

a. How many pairs of vertical angles are formed when three lines (in a plane) intersect at the same point?

b. How many pairs of vertical angles are formed when four lines (in a plane) intersect at the same point?

c. Suppose a fifth line was added to a diagram for Part b and the line intersected the other lines at the same point. How many *additional* pairs of vertical angles are formed? What is the total number of pairs of vertical angles formed by the five lines?

d. Make a conjecture about the number of pairs of vertical angles formed by n lines (in a plane) that intersect at the same point. Assume $n \geq 2$.

10 In Investigation 2, you saw that if you *assume* the Parallel Lines Postulate, then it is possible to prove other conditions on angles that result when parallel lines are cut by a transversal. It is also possible to prove other conditions on angles that guarantee two lines are parallel.

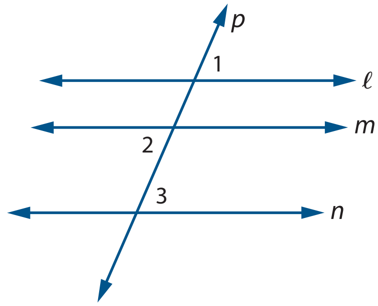
If the Parallel Lines Postulate is rewritten, replacing each occurrence of the phrase "corresponding angles" with "alternate interior angles," the new statement is often called the Alternate Interior Angles Theorem. You proved the two parts of this theorem in Problem 5 on page 37 and in Problem 7 on page 38 using the Parallel Lines Postulate.

Suppose next year's math class *assumes* the Alternate Interior Angles Theorem as its Parallel Lines Postulate.

a. Could the class then *prove* the two parts of the Parallel Lines Postulate? Explain your reasoning.

b. Could the class also prove the relationships between parallelism of lines and angles on the same side of the transversal? Explain.

11 In the diagram below, $\ell \parallel m$ and $m \parallel n$. Line p intersects each of these lines.



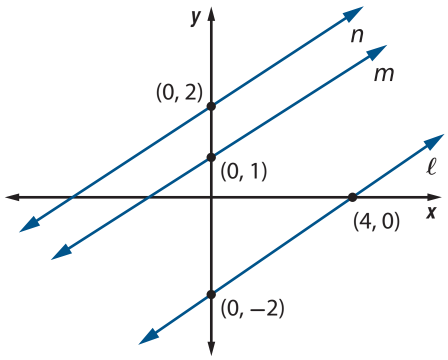
a. How are the measures of angles 1 and 2 related? Explain your reasoning.

b. How are the measures of angles 2 and 3 related? Explain your reasoning.

c. Using your deductions in Parts a and b, prove that lines ℓ and n are parallel.

d. Write an if-then statement that summarizes the theorem you have proved.

12 In the coordinate plane diagram below, lines ℓ, m, and n are parallel and contain the points shown.



a. Write an equation for each line.

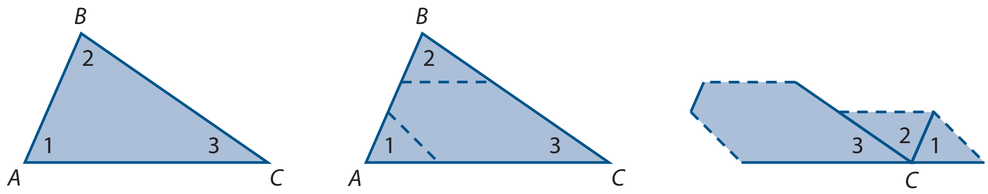
b. On a copy of this diagram, draw a line perpendicular to each line through its y-intercept.

c. How do the three lines you drew in Part b appear to be related to each other?

d. Prove your conjecture in Part c using postulates and/or theorems you proved in Investigation 2.

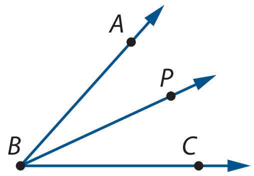
e. Prove your conjecture in Part c using the equations of the lines and coordinate methods.

13 In previous courses, you used the well-known fact that the sum of the measures of the angles of a triangle is 180°. That property was developed through experimentation and inductive reasoning based on several cases. One possible experiment is illustrated below. But how can you be sure this is true for *all* triangles?

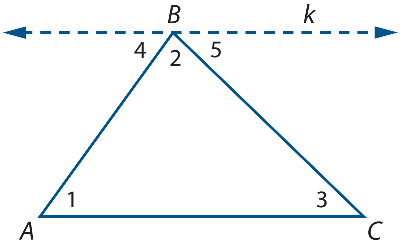


There is an important connection between the Triangle Angle Sum Property and the Alternate Interior Angles Theorem (Connections Task 10) that you proved using the Parallel Lines Postulate. To establish that connection, an additional assumption is needed.

Angle Addition Postulate If P is a point in the interior of $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$.



The diagram below shows a triangle ABC and a line k drawn parallel to $\overleftrightarrow{AC}$ through point B.



a. What construction did you carry out earlier that shows that line k can be drawn parallel to $\overleftrightarrow{AC}$ through point B?

b. How are $\angle 1$ and $\angle 4$ related? How are $\angle 3$ and $\angle 5$ related?

c. Write a deductive proof that $m\angle 1 + m\angle 2 + m\angle 3 = 180°$.

14 In Connections Task 13, you proved that the sum of the measures of the angles of a triangle is 180°. For each of the following statements, decide if it is true. If true, explain why. Otherwise, give a counterexample. (Recall that an acute angle is an angle whose measure is less than 90°.)

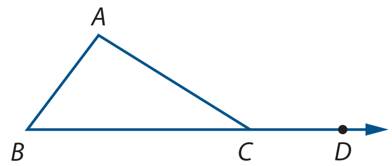
a. If the measures of two angles of one triangle are equal to the measures of two angles of another triangle, then the measures of the third angles are equal.

b. The sum of the measures of the acute angles of a right triangle is 90°.

c. If two angles of a triangle are acute, then the third angle is not acute.

d. If a triangle is equiangular (all angles have the same measure), then each angle has measure 60°.

15 Recall that an *exterior angle* of a triangle is formed when one side of the triangle is extended as shown at the right. $\angle A$ and $\angle B$ are called *remote interior angles* with respect to the exterior angle, $\angle ACD$.



a. How is $m\angle ACD$ related to $m\angle A$ and $m\angle B$?

b. Write an argument to support your claim.

c. Using the term "remote interior angles," write a statement of the theorem you have proved. This theorem is often called the **Exterior Angle Theorem for a Triangle**.

Reflections

16 In this lesson, you proved mathematical statements in geometric settings. How do you decide where to start in constructing a proof? How do you know when the proof is completed?

17 In Investigation 1, you explored how to construct a line perpendicular to a given line through a point on the line and through a point not on the line. In Investigation 2, you discovered a method to construct a line parallel to a given line through a point not on the line. Think about these ideas algebraically in the context of a coordinate plane. Consider the line ℓ with equation $2x - y = 11$.

a. Is $P(8, 5)$ on line ℓ? Write an equation of a line m perpendicular to line ℓ through point P.

b. Write an equation of a line perpendicular to line ℓ through point $Q(8, 0)$.

c. Compare your methods for answering Parts a and b.

d. Write an equation of a line parallel to line ℓ through the point $Q(8, 0)$.

18 Parallel lines were defined to be lines in a plane that do not intersect.

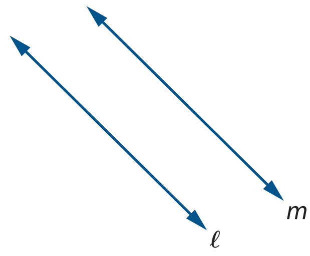
a. Rewrite the definition of parallel lines as an if-and-only-if statement.

b. Is it possible for two lines in three-dimensional space neither to be parallel nor to intersect? Illustrate your reasoning.

19 Determining if edges or lines are parallel is very important in design and construction.

a. Describe at least three methods you could use to test if a pair of lines are parallel.

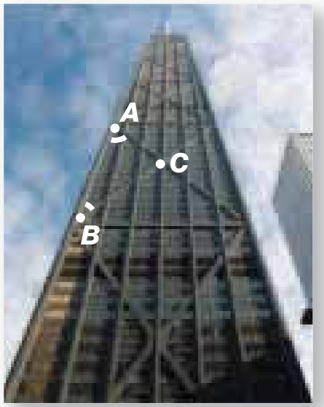
b. Libby claimed that you could determine if two lines are parallel by measuring the perpendicular distance between the lines at two places. Describe how you would perform this test. Provide an argument that justifies Libby's claim.



20 On the John Hancock Center building, are angles of the type marked $\angle ABC$ and $\angle BAC$ congruent or not?

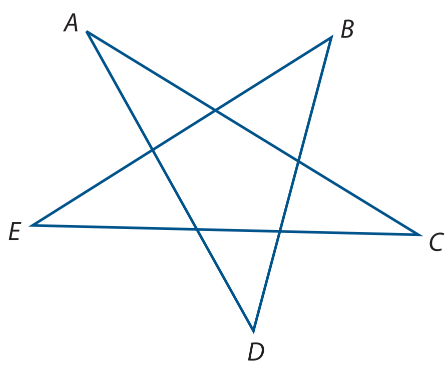
a. Prove your claim.

b. On what assumptions does your proof rely?



Extensions

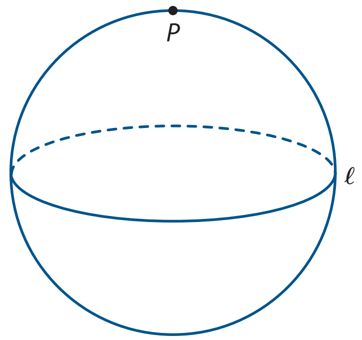
21 What is the sum of the measures of $\angle A$, $\angle B$, $\angle C$, $\angle D$, and $\angle E$? What geometric assumptions and theorems did you use in answering this question?



22 If you look at a map of flight paths of an airline, you will see that the flight paths are not straight lines. Since the Earth is a *sphere*, the shortest path between cities, staying on or slightly above the Earth's surface, follows a **great circle**—a circle on the surface of a sphere formed by a plane passing through the center of the sphere. For example, the equator is a great circle.

Think about how the geometry of a sphere (called *spherical geometry*) differs from the Euclidean geometry of a plane. In spherical geometry, all "lines" (shortest paths on the surface) are great circles.

Consider the equator as line ℓ and the North Pole as point P. In spherical geometry:



a. How many "lines" can be drawn through point P perpendicular to ℓ?

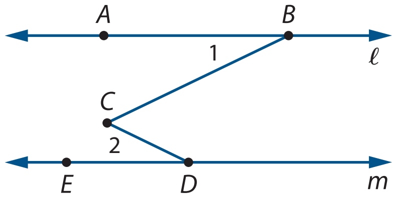
b. How many "lines" can be drawn through point P parallel to ℓ?

c. In spherical geometry, "segments" are parts of great circles. Draw a diagram of a "triangle" PAB for which $m\angle A + m\angle B + m\angle P > 180°$.

d. Draw a diagram to show that the Pythagorean Theorem is not true in spherical geometry.

e. Explain how a "triangle" in spherical geometry can have three right angles.

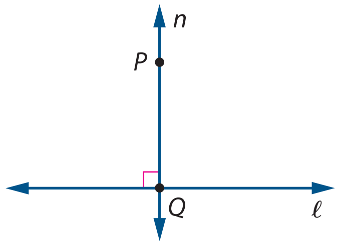
23 In the diagram below, $\ell \parallel m$. Prove that $m\angle BCD = m\angle 1 + m\angle 2$. (*Hint*: Introduce a new line in the diagram as was done in the proof of the Triangle Angle Sum Theorem (page 45). Such a line is called an *auxiliary line*. Its addition to the diagram must be justified by a postulate or previously proved theorem.)



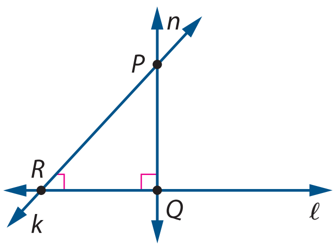
24 Drawing an altitude in a triangle or parallelogram to calculate its area rests on the following theorem.

Through a point not on a given line, there is exactly one line perpendicular to the given line.

A proof of this theorem has two parts. First, establish there is one line n through P perpendicular to ℓ. Then establish there is no more than one line perpendicular to ℓ. You established the first part in Investigation 1, Problem 4 Part c.



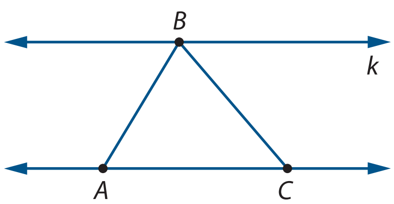
a. Study the following plan for proof of the second part. Give reasons that will support each proposed step.



Plan for Proof:
Use indirect reasoning. Suppose there is a second line k through P perpendicular to ℓ. Show that this will lead to a contradiction. Conclude there is only one line n through P perpendicular to ℓ.

b. How does this theorem compare with the corresponding situation in spherical geometry (Extensions Task 22 Part a)?

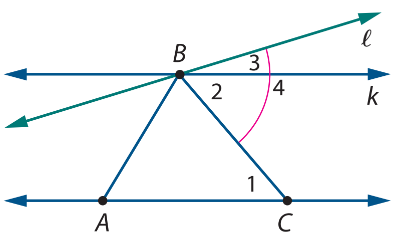
25 In the process of proving the Triangle Angle Sum Theorem (Connections Task 13), you used the previously established fact: Through a point B not on a given line $\overleftrightarrow{AC}$, there is a line k through the point parallel to the given line.



a. Do you think there is more than one line through point B parallel to $\overleftrightarrow{AC}$? Compare your answer with that for Extensions Task 22 Part b.

b. To show there is no more than one parallel line in Part a, you can use *indirect reasoning*.

i. *Assume* there is a second line, call it ℓ, through point B such that $\ell \parallel \overleftrightarrow{AC}$.



- Why is $m\angle 4 = m\angle 1$?
- Why is $m\angle 4 = m\angle 2 + m\angle 3$?
- Why is $m\angle 1 = m\angle 2$?
- Why is $m\angle 4 = m\angle 1 + m\angle 3$?

ii. Identify a *contradiction* in part i.

iii. That contradiction shows the assumption in part i is false. So, what must be true?

Review

26 When trying to solve real-world problems, it is often necessary to write symbolic expressions or rules that indicate relationships described in words. Write a symbolic rule that matches each description. Indicate what your letter symbols represent.

a. There are 3 more cats than dogs in the veterinarian's waiting room.

b. There are 25 students for each teacher on the field trip.

c. There are 3 fewer calculators than there are students in the classroom.

d. There are twice as many sophomores as there are juniors on the organizing committee.

27 The density of an object can be found by dividing the mass of the object by the volume of the object. Using symbols, this can be written $d = \frac{m}{V}$.

a. Rock 1 and Rock 2 have the same volume. The density of Rock 1 is greater than the density of Rock 2. How do their masses compare?

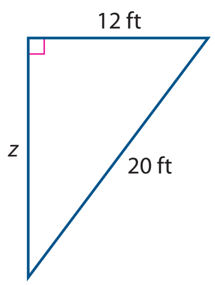
b. Two objects have the same mass but the volume of Object 1 is greater than the volume of Object 2. Compare the densities of the two objects.

c. Rewrite the formula in the form $m = \ldots$.

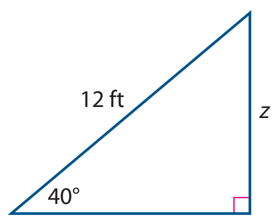
d. Rewrite the formula in the form $V = \ldots$.

28 Use properties of triangles and the right triangle trigonometric ratios to find an approximate value of z in each triangle.

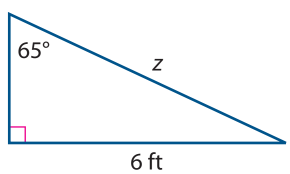
a.



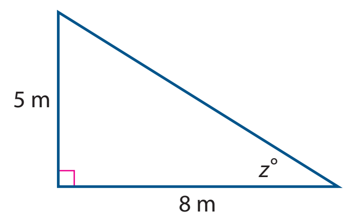
b.



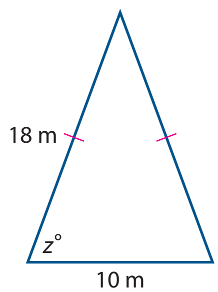
c.



d.



e.



29 Write each of these expressions in equivalent factored form.

a. $3x + 15$ **b.** $5x^2 + 15x$ **c.** $x^3 - x^2$

d. $x^2 - 9$ **e.** $x^2 + 6x + 9$ **f.** $x^2 - 3x - 10$

30 There are 16 marbles in a bag. There are red and blue marbles. If a marble is randomly selected from the bag, the probability of selecting a red one is $\frac{1}{4}$.

a. How many blue marbles are in the bag?

b. Suppose that you reach into the bag, draw one marble, note the color, and return it to the bag. You then draw a second marble from the bag. What is the probability that the two marbles you draw are the same color? Explain your reasoning or show your work.

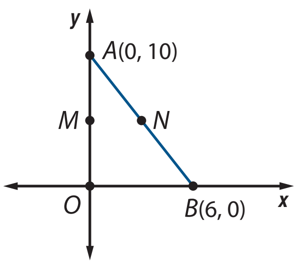
c. How many yellow marbles would you need to add to the bag to make the probability of choosing a red or a yellow marble $\frac{1}{2}$? Explain your reasoning.

31 Write each of these expressions in different equivalent form.

a. $(a^2)(a^5)$ **b.** $\frac{a^5}{a^3}$ **c.** $(a^2)^5$

d. a^{-2} **e.** $(2a^2)^3$ **f.** $a^5(a) - a^5$

32 Refer to the coordinate diagram shown at the right.



a. Find the coordinates of the midpoint M of $\overline{AO}$. The midpoint N of $\overline{AB}$.

b. Use coordinates to explain why $\overline{MN} \parallel \overline{OB}$.

c. How does the length of $\overline{MN}$ compare to the length of $\overline{OB}$?

d. How would your answers for Parts a–c change if point A has coordinates $(0, a)$ and point B has coordinates $(b, 0)$.

33 Solve each equation.

a. $3(2x - 5) = 6 - 2x$ **b.** $5(3^x) = 45$

c. $x(x + 6) = 0$ **d.** $(x + 3)(x - 2) = 6$

34 Determine if the relationship in each table is linear, exponential, or quadratic. Then write a rule that matches each table. Explain your reasoning or show your work.

a.

x	2	3	4	5	6
y	6	11	18	27	38

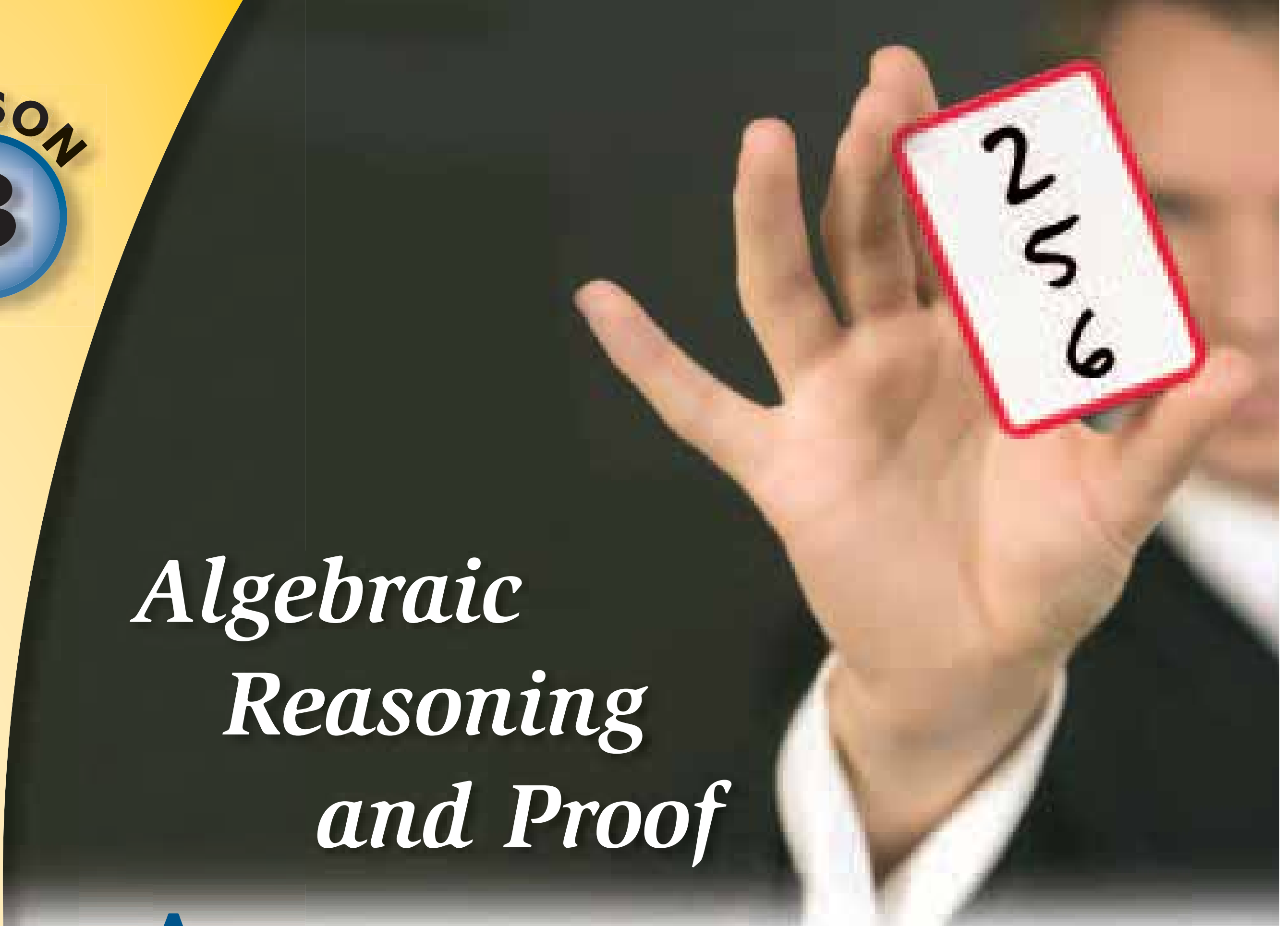
b.

x	4	6	8	10	12
y	2	4.5	7	9.5	12

c.

x	1	2	3	4	5
y	200	100	50	25	12.5

LESSON 3



Algebraic Reasoning and Proof

Algebraic calculations and reasoning can be used to solve many kinds of practical problems. But they can also be used to perform some amazing magic tricks with numbers. For example, do these numerical operations.

> **Pick an integer between 0 and 20.**
> **Add 5 to your number.**
> **Multiply the result by 6.**
> **Divide that result by 3.**
> **Subtract 9 from that result.**

If each class member reports the final number that results from those operations, your teacher will be able to tell the starting number of each student.

If you believe that your teacher performs this number magic by memorizing all the possible starting and ending number combinations, you can increase the range of starting numbers to 0–50 and then 0–100. Your teacher will still be able to find each start number.

Data in the table at the right show some combinations of starting and ending numbers.

Start Number	End Number
4	9
11	23
17	35
33	67
45	91

Think About This Situation

The number magic is not so amazing if you think about it with algebraic reasoning.

a What starting number do you think would lead to an end result of 39? Of 123? Of 513?

b Can you explain how and why your teacher is able to find every starting number when told only the ending number?

In this lesson, you will explore ways to use algebraic reasoning to discover and prove number patterns and relationships like the one used in the number trick. Then you will see how the same reasoning methods can be used to prove other important mathematical principles.

Investigation 1 Reasoning with Algebraic Expressions

In earlier algebra units of *Core-Plus Mathematics*, you developed a toolkit of techniques for writing algebraic expressions in useful equivalent forms. As you work on the problems in this investigation, look for answers to this question:

How can strategies for manipulating algebraic expressions into equivalent forms be used to explain interesting number patterns?

Algebra and Number Magic Algebraic reasoning skills can be used to create and explain many different number tricks.

1 There is a simple way to discover start numbers in the Think About This Situation number trick: *Subtract one from the end number and divide that result by two*. Explaining why that decoding strategy works and proving that it will *always* work requires some algebraic reasoning.

a. Use the letter n to represent the start number and build an algebraic expression that shows the steps for calculating the result that will be reported to the teacher.

i. Adding 5 to your number is expressed by … .

ii. Multiplying that result by 6 is expressed by … .

iii. Dividing that result by 3 is expressed by … .

iv. Subtracting 9 from that result is expressed by … .

b. Use what you know about algebra to write the final expression from Part a in simplest form. What relationship between starting and ending numbers is shown by the result?

c. How can the relationship between starting and ending numbers be used to find starting numbers when only ending numbers are known?

d. Suppose that the third step of the number trick, "Divide that result by 3," is replaced by "Divide that result by 2." How would that change in the procedure affect the decoding strategy?

2 Consider this different number trick.

> **Pick a number.**
>
> **Double it.**
>
> **Add 3 to the result.**
>
> **Multiply that result by 5.**
>
> **Subtract 7 from that result.**

a. Explore the way that this procedure transforms start numbers into final results. Find a decoding strategy that could be used to find the start number when only the final result is known.

b. Use algebraic reasoning to explain why your decoding strategy will always work.

3 Now that you have analyzed some number tricks with algebraic reasoning, you can adapt those arguments to design your own trick.

a. Create a similar number trick. Test it with other students to see if it works as intended.

b. Develop an algebraic argument to prove the trick that you created will work in every case.

Explaining Number Patterns Number patterns have fascinated amateur and professional mathematicians for thousands of years. It is easy to find interesting patterns, but usually more challenging to explain why the patterns work.

4 Consider the sequence of square numbers that begins 0, 1, 4, 9, 16, … .

a. Complete the following table to show how that sequence continues.

n	0	1	2	3	4	5	6	7	8	…	n
S_n	0	1	4	9	16					…	

b. Calculate the differences between consecutive terms in the sequence of square numbers (for example, $S_4 - S_3 = 16 - 9 = 7$). Then describe the pattern that develops in the sequence of differences.

c. If n and $n + 1$ represent consecutive whole numbers, the difference between the squares of those numbers can be given by $S_{n+1} - S_n = (n + 1)^2 - n^2$. Use your ability to simplify algebraic expressions to prove that the pattern of differences you described in Part b is certain to continue as the sequence of square numbers is extended.

d. How would you describe the pattern of differences between successive square numbers as a proposition in the form *If … , then …* ?

5 The pattern that you analyzed in Problem 4 can also be justified with a visual proof.

a. The diagrams shown here have shaded and unshaded regions. Express the area of each unshaded region as a difference of two square numbers.

Diagram I

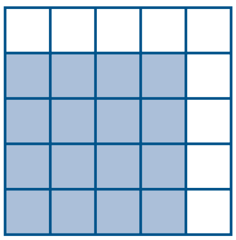
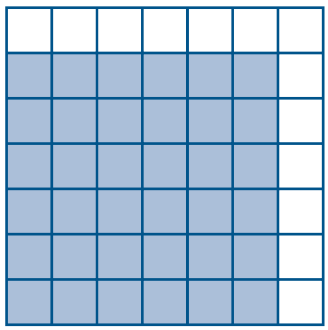
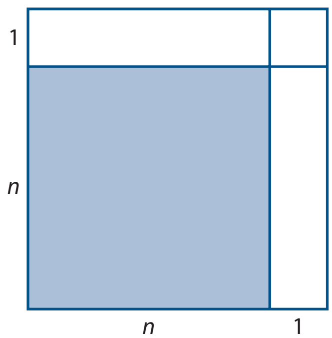


Diagram II



b. Explain how you could use the next diagram to justify the formula for the difference of any two consecutive square numbers that you found in work on Problem 4.



6 Consider the sequence of powers of two that begins 1, 2, 4, 8, 16, … .

a. Complete the following table to show how that sequence continues.

n	0	1	2	3	4	5	6	7	8	…	n
P_n	1	2	4	8	16					…	

b. Calculate the differences between consecutive terms in the sequence P_n. Describe the pattern that develops.

c. What algebraic expression shows how to calculate $P_{n+1} - P_n$?

d. Use the result from Part c, what you know about writing exponential expressions in equivalent forms, and other algebraic manipulations to prove that the pattern observed in Part b is certain to continue as the sequence is extended.

e. How would you describe the pattern of differences between successive powers of 2 as a proposition in the form *If* … , *then* … ?

Proving Properties of Numbers and Functions In the Course 2 unit, *Nonlinear Functions and Equations*, you learned how to use *logarithms* to describe the intensity of sound and earthquakes and the pH of liquids. Historically, however, logarithms were first used as an aid to scientific calculation involving large numbers. That application of logarithms depends on a very useful property of the function $y = \log x$.

7 Recall that $y = \log x$ means $10^y = x$. For example, $3 = \log 1{,}000$ because $10^3 = 1{,}000$. Use reasoning about powers of 10 or the **log** function on your calculator to complete the following table that compares $\log a$, $\log b$, and $\log ab$ for a sample of positive numbers a and b.

a	b	ab	$\log a$	$\log b$	$\log ab$
100	1,000				
100	0.001				
0.1	0.001				
75	100				
75	0.01				
75	120				
50	20				

What pattern do you see that relates $\log a$, $\log b$, and $\log ab$? Describe the pattern in words and algebraically with symbols.

8 Rows in the table of Problem 7 illustrate a property of logarithms that connects multiplication and addition in a very useful way: For any a and $b > 0$, $\log ab = \log a + \log b$. Use what you know about logarithms and exponents to justify each step in the following algebraic proof of that property.

(1) Since $a > 0$ and $b > 0$, there are numbers x and y so that $x = \log a$ and $y = \log b$.
(2) So, $a = 10^x$ and $b = 10^y$.
(3) $\log ab = \log (10^x 10^y)$
(4) $\qquad = \log 10^{x+y}$
(5) $\qquad = x + y$
(6) $\qquad = \log a + \log b$
(7) Therefore, $\log ab = \log a + \log b$.

This property of the logarithm function enabled mathematicians and scientists to use a table giving the logarithms of many numbers to convert any multiplication problem into an easier addition problem. This strategy was particularly useful for calculations involving very large numbers, such as those that occur in sciences, because it is much simpler to add logarithms of two large numbers than to find the product of the numbers.

9 The speed of light travels at about 186,000 miles per second, and there are 31,536,000 seconds in one year. To find the length in miles of a *light year*, you need to multiply 186,000 × 31,536,000.

a. To calculate, apply the standard multiplication algorithm.

$$\begin{array}{r} 31{,}536{,}000 \\ \times\ 186{,}000 \\ \hline \end{array}$$

b. Use the facts $\log 186{,}000 \approx 5.26951$ and $\log 31{,}536{,}000 \approx 7.49881$ to find $\log (186{,}000 \times 31{,}536{,}000)$. Then use the definition $y = \log x$ when $10^y = x$ to find $186{,}000 \times 31{,}536{,}000$.

c. Which procedure for finding the product of large numbers involves the simplest arithmetic?

Summarize the Mathematics

In this investigation, you explored ways that algebraic reasoning explains interesting number patterns.

a. What were the key steps in explaining the number tricks discovered in Problems 1 and 2?

b. What overall strategy and algebraic properties were used to prove the generality of number patterns discovered in Problems 4 and 6?

c. What overall strategy and algebraic properties were used to prove that $\log ab = \log a + \log b$ for all positive values of a and b?

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of equivalent expressions and algebraic reasoning to complete these tasks.

a. Prove that, whatever the starting number, the following calculations always lead to a result that is one greater than the starting number.

> **Think of a number.**
>
> **Double that number and add 7.**
>
> **Multiply that result by 5 and subtract 25.**
>
> **Divide that result by 10.**

b. Consider the sequence $S_n = (n - 1)^2 + 1$.

i. Write the first seven terms of this sequence.

ii. Calculate the differences of consecutive terms in that sequence and describe the pattern that develops.

iii. Use algebraic reasoning to prove that the pattern you described in part ii is certain to continue as the number sequence is extended.

CPMP-Tools

Investigation 2 Reasoning with Algebraic Equations

Many important mathematical facts are expressed in the form of equations or inequalities relating two or more variables. For example, the coordinates (x, y) of points on a line will always be related by an equation in the form $ax + by = c$. In this investigation, look for answers to this question:

How can strategies for reasoning with algebraic equations be used to explain and prove important principles in algebra, geometry, and trigonometry?

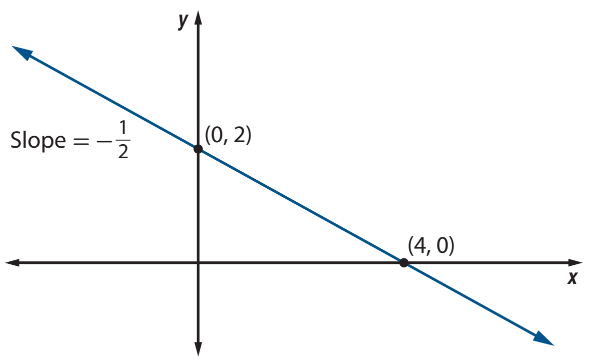
1 Mathematical work often requires solving linear equations, so it might be convenient to write a short calculator or computer program that will perform the necessary steps quickly when given only the equation coefficients and constants.

a. What formula shows how to use the values of a, b, and c to find the solution for any linear equation in the form $ax + b = c$ with $a \neq 0$? Show the algebraic reasoning used to derive the formula. Check your work by asking a computer algebra system (CAS) to **solve(a*x+b=c,x)** and reconcile any apparent differences in results.

b. What formula shows how to use the values of a, b, c, and d to find the solution for any linear equation in the form $ax + b = cx + d$ with $a \neq c$? Show the algebraic reasoning used to derive the formula. Check your work by asking a CAS to **solve(a*x+b=c*x+d,x)** and reconcile any apparent differences in results.

c. What statements would express the formulas of Parts a and b in if-then form?

2 When a problem requires analysis of a linear equation in two variables, it often helps to draw a graph of the equation. Information about x- and y-intercepts and slope can be used to make a quick sketch. For example, the diagram at the right shows the slope and intercepts of the graph for solutions of the equation $3x + 6y = 12$.



Find the x- and y-intercepts and the slopes for lines with equations given in Parts a–d. Use that information to *sketch* graphs of the equations, labeling the x- and y-intercepts with their coordinates.

a. $-6x + 4y = 12$ **b.** $6x + 3y = 15$

c. $5x - 3y = -5$ **d.** $4y + 5x = -8$

3 To find formulas that give intercepts and slope of any line with equation in the form $ax + by = c$, you might begin by reasoning as follows.

(1) The y-intercept point has coordinates in the form $(0, y)$ for some value of y.

(2) If $a(0) + by = c$, then $y = \ldots$.

a. Complete the reasoning to find a formula for calculating coordinates of the y-intercept. Describe any constraints on a and b.

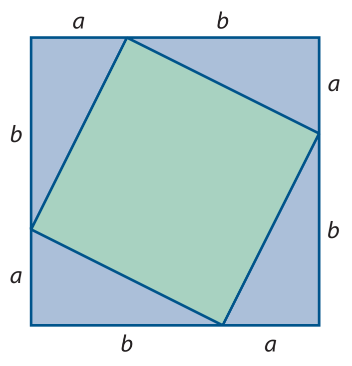
b. Use reasoning like that in Part a to find a formula for calculating coordinates of the x-intercept of any line with equation in the form $ax + by = c$. Describe any constraints on a and b.

c. Use reasoning about equations to find a formula for calculating the slope of any line with equation in the form $ax + by = c$ with $b \neq 0$.

d. Write CAS solve commands that could be used to check your work in Parts a, b, and c.

4 You probably recall the Pythagorean Theorem that relates lengths of the legs and the hypotenuse in any right triangle. It says, "If a and b are lengths of the legs and c the length of the hypotenuse of a right triangle, then $a^2 + b^2 = c^2$."

There are over 300 different proofs of the Pythagorean Theorem. In Course 1, you completed a proof of the Pythagorean Theorem that was based on a comparison of areas. The diagram at the right shows how to start on a proof that makes use of geometric and algebraic reasoning in important ways. It shows a large square divided into four triangles and a smaller quadrilateral inside that square.



a. Use only the fact that the whole colored figure is a square and the division of its edges into lengths a and b to prove that the four triangles are congruent right triangles.

b. How can you use the fact that the outside figures are congruent right triangles to prove that the interior quadrilateral is a square?

c. Write two different algebraic expressions that show how to find the area of the large square.

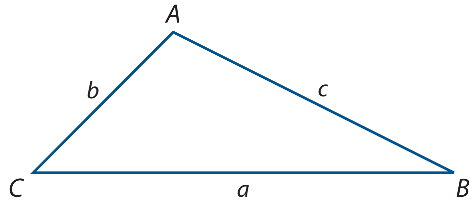
- In the first, use the fact that the sides of the large square are of length $(a + b)$.
- In the second, label the edges of the inner square with length c. Use the fact that the large square is made up of four triangles, each with base a and height b, and an inner square with sides of length c.

d. Equate the two expressions from Part b and apply algebraic reasoning to that equation in order to show that $a^2 + b^2 = c^2$.

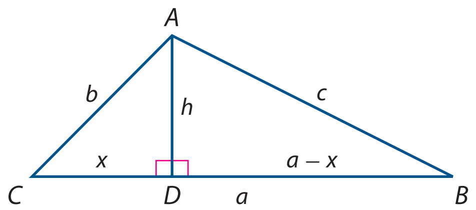
5 The kind of algebraic reasoning used to prove the Pythagorean Theorem can be adapted to prove the very useful Law of Cosines, introduced in Course 2.

In any triangle ABC with sides of length a, b, and c opposite $\angle A$, $\angle B$, and $\angle C$, respectively,

$c^2 = a^2 + b^2 - 2ab \cos C.$



Analyze the following argument to identify and explain ways that algebraic principles (and a small amount of geometry and trigonometry) can be used to prove the special case pictured. Here $\angle C$ is an acute angle and $\overline{AD}$ is an altitude of the triangle with point D between points C and B.



Step 1. In $\triangle ADC$: $\frac{x}{b} = \cos C$, so $x = b \cos C$.
Why are these relationships true?

Step 2. In $\triangle ABD$: $h^2 = c^2 - (a - x)^2$
$= c^2 - (a^2 - 2ax + x^2)$
$= c^2 - a^2 + 2ax - x^2$
Why are these equations true?

Step 3. In $\triangle ADC$: $h^2 = b^2 - x^2$
Why is this equation true?

Step 4. From Steps 2 and 3, we can conclude that
$c^2 - a^2 + 2ax - x^2 = b^2 - x^2$.
Why is this equation true?

Step 5. From the equation in Step 4, we can conclude that
$c^2 = a^2 + b^2 - 2ax$.
What rules of algebra justify this conclusion?

Step 6. Finally, combining Steps 1 and 5, we conclude that
$c^2 = a^2 + b^2 - 2ab \cos C$.
What property justifies this conclusion?

You can check that the same formula results in the cases where point D coincides with or is to the right of point B and that, as long as $\angle C$ is acute, point D cannot lie to the left of point C. A complete proof of the Law of Cosines requires considering cases when $\angle C$ is a right angle or an obtuse angle. See Reflections Task 24.

Summarize the Mathematics

In this investigation, you explored ways that reasoning with equations can be used to prove important general principles in algebra, geometry, and trigonometry.

a What general properties of numbers, operations, and equations did you use to discover and prove a formula for solving equations in the form $ax + b = c$ with $a \neq 0$? In the form $ax + b = cx + d$ with $a \neq c$?

b What general properties of numbers, operations, and equations did you use to discover and prove formulas for slope and intercepts of graphs for linear equations in the form $ax + by = c$?

c What algebraic principles were used to justify steps in the proof of the Pythagorean Theorem?

d What is the main idea behind the proof of the Law of Cosines?

Be prepared to explain your ideas to the class.

Check Your Understanding

CPMP-Tools

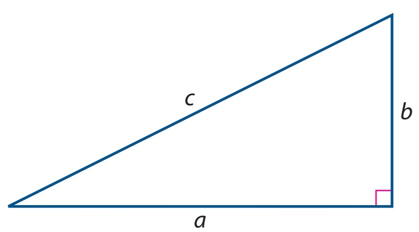
Use your understanding of equations and algebraic reasoning to complete these tasks.

a. Consider linear equations that occur in the form $a(x - b) = c$.

i. Solve the particular example $3(x - 7) = 12$.

ii. Write a formula that will give solutions to any equation in the form $a(x - b) = c$ with $a \neq 0$, in terms of a, b, and c. Show the algebraic reasoning used to derive your formula. Check your work by using a computer algebra system to **solve(a*(x–b)=c,x)**.

b. Many problems in geometry require finding the length of one leg in a right triangle when given information about the lengths of the other leg and the hypotenuse.



i. If the hypotenuse of a right triangle is 20 inches long and one leg is 15 inches long, what is the length of the other leg? Show the reasoning that leads to your answer.

ii. Starting with the algebraic statement of the Pythagorean Theorem $a^2 + b^2 = c^2$, use algebraic reasoning to derive a formula that shows how to calculate the length a when lengths b and c are known.

On Your Own

Applications

1 Whenever three consecutive integers are added together, the sum is always equal to three times the middle number. For example, $7 + 8 + 9 = 3(8)$ and $-4 + -3 + -2 = 3(-3)$.

a. If n represents the smallest number in a sequence of three consecutive integers, what expressions represent:

i. the other two numbers?

ii. the sum of all three numbers?

b. Simplify the expression for the sum in Part a to prove that the pattern relating the sum and the middle number occurs whenever three consecutive integers are added.

c. What pattern will relate the sum of any five consecutive integers to one of the integer addends? Use algebraic reasoning to prove your idea.

2 By definition, any odd number can be expressed in the form $2m + 1$ for some integer m. Any even number can be expressed in the form $2m$ for some integer m.

a. Express 5, 17, and 231 in the form $2m + 1$.

b. Express 6, 18, and 94 in the form $2m$.

c. Prove that the product of an odd number and an even number is always an even number. Begin by writing the odd number as $2m + 1$ and the even number as $2n$.

d. Prove that the product of any two odd numbers is always an odd number. Use different letters in representing the two numbers symbolically.

e. Why are different letters, m and n, used to represent integers in Parts c and d?

3 The two-digit number 53 can be written in *expanded form* as $5(10) + 3$. In general, a two-digit number with tens digit a and ones digit b can be written as $a(10) + b$. Use this fact to prove the number patterns described below.

a. Pick any two-digit number and reverse the order of the digits. Then add the new number to the original number. The result will always be a multiple of 11.

b. Pick any two-digit number and reverse the order of the digits. Then find the difference between the new number and the original number. The result will always be a multiple of 9.

4 Consider the sequence 1, 3, 9, 27, 81, 243, 729, … generated by powers of 3. Use reasoning similar to your work in Problem 6 of Investigation 1 to prove that the differences between successive terms of this sequence can be calculated using the expression $2(3^n)$.

5 The beginnings of two number sequences are given here.

Sequence I 1, 4, 16, 64, 256, 1,024, 4,096, …
Sequence II 1, 5, 25, 125, 625, 3,125, 15,625, …

For each sequence:

a. Find an algebraic rule that shows how to calculate the nth term in the sequence.

b. Study the pattern of differences between successive terms and find a formula that gives the nth difference.

c. Prove that your formula is correct.

6 If the logarithm of the product of two positive numbers is the sum of the logarithms of the individual numbers, you might expect that the logarithm of the quotient of the two numbers is the difference of their logarithms. Use reasoning about powers of 10 or the **log** function of your calculator to check the property $\log\left(\frac{a}{b}\right) = \log a - \log b$ for the sample of positive numbers a and b in this table.

a	b	$\frac{a}{b}$	$\log a$	$\log b$	$\log\left(\frac{a}{b}\right)$
1,000	10				
0.01	100.01				
0.001	0.1				
75	100				
75	0.01				
75	3				

7 Adapt the reasoning in Problem 8 of Investigation 1 to prove that for any positive numbers a and b, $\log\left(\frac{a}{b}\right) = \log a - \log b$. (*Hint*: Recall that $\left(\frac{1}{10}\right)^y = 10^{-y}$.)

8 Consider linear equations that occur in the form $a(x + b) = c$ with $a \neq 0$.

a. Solve the particular example $7(x + 3) = 49$.

b. Write a formula that will give solutions for any equation in the form $a(x + b) = c$ with $a \neq 0$, in terms of a, b, and c. Show the algebraic reasoning used to derive the formula.

9 If a line has slope m and passes through the point (p, q), it is sometimes convenient to write the equation for that line in the form $(y - q) = m(x - p)$.

a. Write an equation in this form for the line that has slope 2 and passes through the point (4, 7).

- Check that coordinates of the points (4, 7) and (5, 9) satisfy your equation.
- Use the two points to check that the slope of the graph will be 2.

b. Explain how you know that the point (p, q) is always on the graph of $(y - q) = m(x - p)$.

c. Use algebraic reasoning to write the equation from Part a in $y = mx + b$ form.

d. Show how any equation in the form $(y - q) = m(x - p)$ can be written in equivalent $y = mx + b$ form.

e. Look back at your work in Part d. How is b related to p and q?

10 You have seen in the problems of this lesson that algebraic reasoning usually involves writing symbolic expressions or equations in equivalent forms using rules for symbol manipulation. The most useful symbol manipulations are based on the list of algebraic properties of equality and operations shown in the following charts.

Algebraic Properties

Addition

For any numbers a, b, and c:

Commutative Property of Addition:	$a + b = b + a$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$
Additive identity element (0):	$a + 0 = a$
Additive Inverse Property:	There is a number $-a$ such that $a + (-a) = 0$.

Multiplication

For any numbers a, b, and c:

Commutative Property of Multiplication:	$ab = ba$
Associative Property of Multiplication:	$(ab)c = a(bc)$
Multiplicative identity element (1):	$a1 = a$
Multiplicative Inverse Property:	For each $a \neq 0$, there is a number a^{-1} such that $a^{-1}a = 1$.

Distributive Property of Multiplication over Addition

For any numbers a, b, and c:	$a(b + c) = ab + ac$

Properties of Equality

For any numbers a, b, and c:

Transitive Property of Equality:	If $a = b$ and $b = c$, then $a = c$.
Addition Property of Equality:	If $a = b$, then $a + c = b + c$.
Subtraction Property of Equality:	If $a = b$, then $a - c = b - c$.
Multiplication Property of Equality:	If $a = b$, then $ac = bc$.
Division Property of Equality:	If $a = b$, then $a \div c = b \div c$ (whenever $c \neq 0$).

State the properties that justify steps in the following derivations of equivalent expressions. Some steps may involve more than one property. Arithmetic and substitution may also be used to justify steps.

a. For any x, $3x + 5x = (3 + 5)x$ (1)

$= 8x$ (2)

b. For any x, $(3x + 5) + 7x = (5 + 3x) + 7x$ (1)

$= 5 + (3x + 7x)$ (2)

$= 5 + 10x$ (3)

c. If $7x + 5 = 5x + 14$, then $(7x + 5) + (-5) = (5x + 14) + (-5)$ (1)

$7x + (5 + (-5)) = 5x + (14 + (-5))$ (2)

$7x + 0 = 5x + 9$ (3)

$7x = 5x + 9$ (4)

$-5x + 7x = -5x + (5x + 9)$ (5)

$2x = (-5x + 5x) + 9$ (6)

$2x = 0x + 9$ (7)

$2x = 9$ (8)

$\left(\frac{1}{2}\right)(2)x = \left(\frac{1}{2}\right)9$ (9)

$1x = 4.5$ (10)

$x = 4.5$ (11)

11 In your work on Task 10 Part c, you probably used a property of multiplication that seemed so obvious it did not need justification: For any n, $0n = 0$. While not stated as one of the basic number system properties, this result can be proven from the listed properties. Provide explanations for each step in the following proof.

$0 + 0 = 0$ (1)

$(0 + 0)n = 0n$ (2)

$0n + 0n = 0n$ (3)

$0n + 0n = 0n + 0$ (4)

$0n = 0$ (5)

12 In your previous coursework, you used the **Zero Product Property**:

If $ab = 0$, then $a = 0$ or $b = 0$.

a. Show how this property is used to solve $(x + 6)(2x - 8) = 0$.

b. Use properties listed in Tasks 10 and 11 to justify each step in the following proof of the Zero Product Property.

Suppose that $ab = 0$. If $a = 0$, the result is clearly true. Suppose $a \neq 0$.

(1) Then there is a number a^{-1} so that $a^{-1}a = 1$.

(2) $a^{-1}(ab) = a^{-1}0$

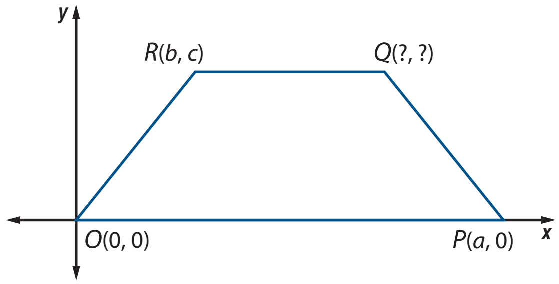
(3) $(a^{-1}a)b = 0$

(4) $1b = 0$

(5) $b = 0$

c. What is the logic of the argument given in Part b to prove the claim that "if $ab = 0$, then $a = 0$ or $b = 0$"?

13 An **isosceles trapezoid** is shown in the coordinate diagram below. It has a pair of nonparallel opposite sides the same length. Use ideas from geometry and trigonometry and algebraic reasoning to complete the following tasks.



a. Draw and label a similar diagram on a coordinate grid. Determine the coordinates of point Q.

b. Prove that the *base angles*, $\angle O$ and $\angle P$, have the same measure.

c. Draw diagonals $\overline{OQ}$ and $\overline{PR}$. How do the diagonals appear to be related? Prove your conjecture.

d. Find the coordinates of the midpoint M of $\overline{OR}$ and the midpoint N of $\overline{PQ}$. Draw $\overline{MN}$. How does $\overline{MN}$ appear to be related to $\overline{OP}$? Prove your conjecture.

e. Write three general statements that summarize what you have proven about isosceles trapezoids.

14 Because proportional reasoning is required to solve many practical, scientific, and mathematical problems, it is important to be able to solve proportions in specific and general cases.

a. Solve these proportions for x.

i. $\frac{3}{x} = \frac{9}{15}$ **ii.** $\frac{x}{12} = \frac{10}{30}$ **iii.** $\frac{4}{5} = \frac{10}{x}$

b. What algebraic reasoning justifies the fact that for any a, b, and c ($a \neq 0$, $b \neq 0$, and $x \neq 0$) if $\frac{a}{b} = \frac{c}{x}$, then $x = \frac{bc}{a}$?

Connections

15 The kind of algebraic reasoning you used to explain and create number tricks can also be used to explain some useful procedures for rewriting algebraic expressions in convenient equivalent forms. For example, you might impress friends or parents with your mental arithmetic skills by backing up the following claim.

> "Pick two numbers that are equidistant from 100, like 112 and 88, and I'll find their product almost immediately without using a calculator."

Do you know how to do this kind of calculation in your head quickly? You can use the fact that for any two numbers r and s, the product $(r + s)(r - s) = r^2 - s^2$.

a. Explain why each step in the following algebraic proof is justified.

$$\begin{aligned}(r + s)(r - s) &= (r + s)r - (r + s)s && (1)\\ &= r^2 + sr - rs - s^2 && (2)\\ &= r^2 + rs - rs - s^2 && (3)\\ &= r^2 - s^2 && (4)\end{aligned}$$

b. How could the given algebraic relationship be used to calculate 105×95 or 112×88?

c. How could you quickly calculate products like $990 \times 1{,}010$ or $996 \times 1{,}004$?

16 The expanded form of any expression like $(x + p)^2$ is $x^2 + 2px + p^2$.

a. Check this relationship by calculating the value of each expression when:

i. $x = 3$ and $p = 7$

ii. $x = 3$ and $p = -7$

iii. $x = -4$ and $p = -3$

b. Adapt the reasoning in Task 15 to develop an algebraic proof of expanding perfect squares, $(x + p)^2 = x^2 + 2px + p^2$.

17 Suppose that in one school term, you have five mathematics quiz scores x_1, x_2, x_3, x_4, and x_5.

a. What algebraic expression shows how to calculate the mean of those scores?

b. Use algebraic reasoning to show how the mean of your quiz scores is affected if each score is multiplied by some common factor p.

c. Use algebraic reasoning to show how the mean of your quiz scores is affected if a constant k is added to each score.

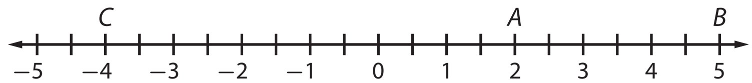
18 Suppose that $\bar{x}$ is the mean of five mathematics quiz scores x_1, x_2, x_3, x_4, and x_5.

a. Describe in words how to calculate the standard deviation $s = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n}}$ for this set of scores.

b. Use algebraic reasoning to show how the standard deviation is affected if each score is multiplied by some common factor p.

c. Use algebraic reasoning to show how the standard deviation is affected if a constant k is added to each score.

19 Recall that for any two points on a number line, the distance between them is the absolute value of the difference of their coordinates. For example, on the number line below, the distance between points A and B is given by $|5 - 2| = |2 - 5| = 3$, and the distance between points B and C is given by $|5 - (-4)| = |(-4) - 5| = 9$.



a. Find the numbers represented by the following absolute value expressions.

i. $|12 - 3|$ **ii.** $|3 - 12|$ **iii.** $|7 - (-4)|$

iv. $|(-4) - 7|$ **v.** $|(-7) - (-3)|$ **vi.** $|(-3) - (-7)|$

b. What can you conclude about the relationship between numbers p and q in each case below?

i. $|p - q| = p - q$ **ii.** $|p - q| = q - p$ **iii.** $|p - q| = 0$

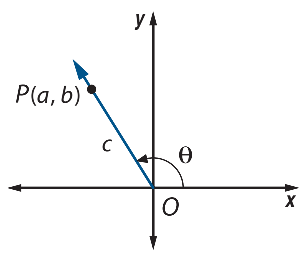
c. Use your answers to Part b to prove that the following calculations can be used to compare any two distinct numbers p and q.

i. The larger of numbers p and q will always be given by $\frac{p + q + |p - q|}{2}$.

ii. The smaller of numbers p and q will always be given by $\frac{p + q - |p - q|}{2}$.

20 Algebraic reasoning can be used to prove one of the most useful relationships in trigonometry. Justify each step in the following proof that for any angle θ, $\sin^2 \theta + \cos^2 \theta = 1$. The notation $\sin^2 \theta$ means $(\sin \theta)^2$.

Proof: Suppose θ is an angle in standard position in the coordinate plane, and $P(a, b)$ is a point on the terminal side of θ. Let c be the distance from the origin to point P.



Then:

$$\sin \theta = \frac{b}{c} \text{ and } \cos \theta = \frac{a}{c} \qquad (1)$$

$$\sin^2 \theta = \frac{b^2}{c^2} \text{ and } \cos^2 \theta = \frac{a^2}{c^2} \qquad (2)$$

$$\sin^2 \theta + \cos^2 \theta = \frac{b^2}{c^2} + \frac{a^2}{c^2} \qquad (3)$$

$$\sin^2 \theta + \cos^2 \theta = \frac{b^2 + a^2}{c^2} \qquad (4)$$

$$\sin^2 \theta + \cos^2 \theta = \frac{c^2}{c^2} \qquad (5)$$

$$\sin^2 \theta + \cos^2 \theta = 1 \qquad (6)$$

Reflections

21 Students in one Colorado school came up with a simple rule of thumb for deciding on the kinds of algebraic manipulations that would transform given expressions into forms that they could be sure were equivalent to the original. They reasoned, "You can do anything to algebraic expressions that you can do to expressions that involve only specific numbers." Does this seem like a safe guideline for symbol manipulation?

22 A mathematician once said that there are three stages in proving the truth of some mathematical proposition. First, you have to convince yourself. Second, you have to convince your friends. Third, you have to convince your enemies. The kinds of evidence that you find convincing for yourself might be different than the evidence that will convince someone else.

a. What kinds of evidence do you find to be most persuasive: a collection of specific examples, a visual image of the pattern, or an algebraic argument that links given conditions to conclusions by a sequence of equivalent equations or expressions?

b. Describe the value of each of these types of persuasive arguments.

i. several specific illustrative examples

ii. a visual image

iii. formal algebraic reasoning

23 In Lesson 2, you used geometric reasoning to prove that if two lines in a plane are perpendicular to the same line, then they are parallel to each other. How could you prove that statement using algebraic reasoning? Assume ℓ, p, and m are lines with $\ell \perp p$, $m \perp p$, and the slope of ℓ is a where $a \neq 0$.

24 In Investigation 2, you completed a proof of the Law of Cosines for one case of an acute angle in a triangle.

a. Why is the Law of Cosines true for the case where $\angle C$ is a right angle?

b. Modify the diagram and its labeling on page 60 for the case where $\angle C$ is an obtuse angle. How would you modify the argument in the text for this case?

25 How is reasoning with algebraic properties as in Applications Tasks 10–12 similar to reasoning with postulates in geometry?

Extensions

26 A number trick found on the Web site www.flashpsychic.com begins with these directions.

Pick a two-digit number.

Find the sum of the digits.

Subtract that digit sum from the original number.

Then the program asks you to find your result in a table that pairs each two-digit number with a special symbol. The program proceeds to tell you the symbol corresponding to your ending number.

a. Visit the Web site and see if the program works as advertised for you.

b. Use algebraic reasoning to explain how the number trick works.

27 The following table shows days and dates for January 2010. Several sets of four related dates have been enclosed in boxes.

Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

a. In each box, identify the pairs of date numbers that are at opposite corners. Find the products of those pairs, then find the differences of the products.

b. Describe the pattern relating all results of the calculations prescribed in Part a and check some more 2 × 2 boxes in the calendar page to test your pattern conjecture.

c. If the number in the upper-left corner of a box is x, how can the other three numbers in the box be expressed in terms of x?

d. Use the expressions in Part c and algebraic reasoning to prove the pattern that you described in Part b will hold for any such 2 × 2 box of dates on any calendar page arranged in rows like that above.

28 Explore the output of the following algorithm.

Start with the first three digits of your phone number (not the area code).

Multiply that entry by 80 and add 1.

Multiply that result by 250.

Double the number given by the last four digits of your phone number and add that number to your result from the previous step.

Subtract 250 from that result.

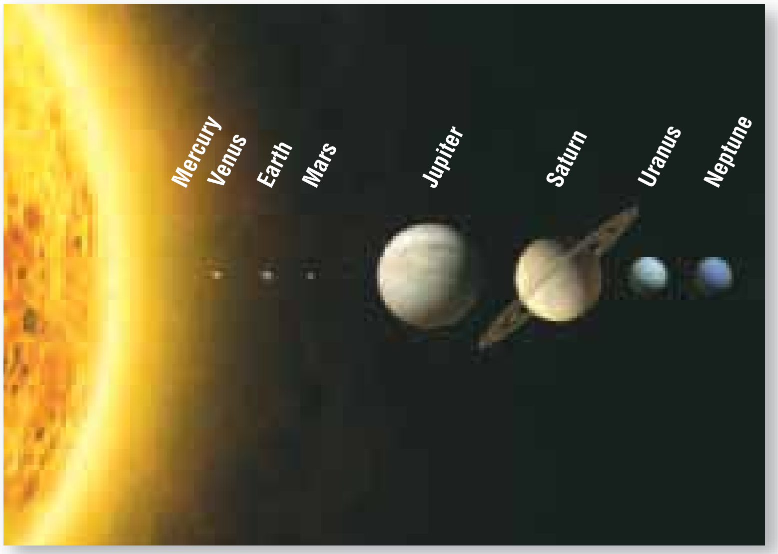
Divide that result by 2.

a. What does the algorithm seem to do with your phone number?

b. Use x to represent the number defined by the first three digits of a phone number and y to represent the number defined by the last four digits of a phone number and write an expression that shows how the algorithm operates.

c. Simplify the expression in Part b to explain the connection between the entry and result of the algorithm.

29 The largest planet in our solar system, Jupiter, has volume of about 333,000,000,000,000 cubic miles, while our Earth has volume of about 268,000,000,000 cubic miles.



a. Explain why log 333,000,000,000,000 = 14 + log 3.33. Use a calculator or software **log** function to approximate this value.

b. Explain why log 268,000,000,000 = 11 + log 2.68. Use a calculator or software **log** function to approximate this value.

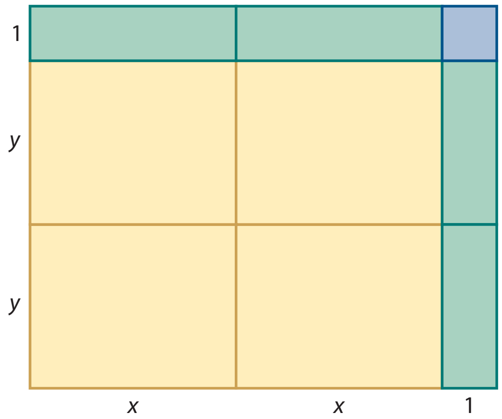
c. Use the results of Parts a and b to approximate log (333,000,000,000,000 ÷ 268,000,000,000).

d. Use the result of Part c and the fact that $y = \log x$ when $10^y = x$ to find an approximate value of the quotient 333,000,000,000,000 ÷ 268,000,000,000. Then explain what the result tells about the ratio of the volume of Jupiter to the volume of Earth.

e. Explain why 333,000,000,000,000 ÷ 268,000,000,000 = 333,000 ÷ 268. Find that quotient and compare the result to your answer in Part d.

f. The average radius of Jupiter is about 43,000 miles and that of Earth is about 4,000 miles. What do those figures imply about the ratio of the volumes of the two planets, and how does your answer compare to that in Part d?

30 Explain how the diagram below gives a visual "proof without words" of the fact that the product of two odd numbers is always an odd number.



Review

31 Use algebraic reasoning to solve the following equations for x.

a. $12x - 9 = -7x + 6$

b. $3x^2 + 7 = 55$

c. $3x^2 + 7x = 0$

d. $x^2 + 5x - 14 = 0$

e. $4(x - 5) = 5 + 7x$

32 Use properties of exponents to write each of the following expressions in equivalent simplest form, using only positive exponents.

a. p^3p^7

b. $(p^3)^2$

c. $(p^2q^3)(p^{-1}q^2)$

d. $\frac{p^4q^3}{q^2p^2}$

e. $\left(\frac{p^2q}{pq^3}\right)^4$

33 A is the point with coordinates $(6, 3)$, B is the point $(-4, 8)$, and O is the origin $(0, 0)$.

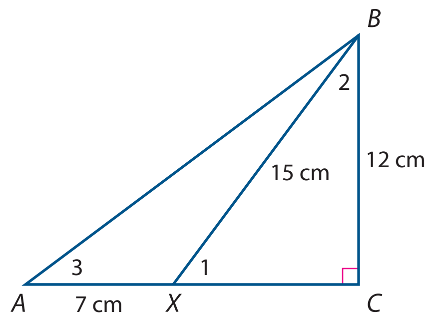
a. Plot these points in the coordinate plane. Draw $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$. How do these two lines appear to be related?

b. Justify your conjecture in Part a in two different ways.

i. using slopes

ii. using the distance formula

34 In the diagram at the right, $\angle C$ is a right angle. Use information given to find the requested trigonometric function values. Express your answers as fractions in simplest form.



a. $\sin \angle 1$ **b.** $\cos \angle 1$

c. $\cos \angle 2$ **d.** $\sin \angle 2$

e. $\sin \angle 3$ **f.** $\cos \angle 3$

g. $\tan \angle 1$ **h.** $\tan \angle 2$

i. $\tan \angle 3$

35 Suppose that in the diagram of Task 34, $m\angle 1$ is 53° and $m\angle 3$ is 37°. Find the measures of these angles:

a. $\angle 2$

b. $\angle AXB$

c. $\angle XBA$

36 Find algebraic rules for functions with these properties.

a. A linear function $f(x)$ for which $f(0) = -5$ and $f(4) = 7$

b. A linear function $g(x)$ for which $g(3) = -5$ and $g(-5) = -1$

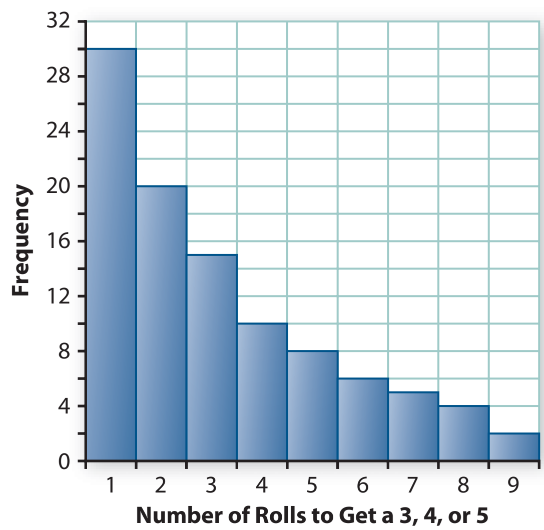
c. A quadratic function $h(x)$ for which $h(0) = 4$, $h(4) = 0$, and $h(1) = 0$

d. An exponential function $j(x)$ for which $j(0) = 5$ and $j(1) = 15$

37 Keisha has created a game in which the player rolls two dice and must roll a sum of 3, 4, or 5 in order to move a gameboard piece into play.

a. What is the probability that a person will be able to move a piece into play on the first roll?

b. While testing the game, Keisha rolled the dice until she got a sum of 3, 4, or 5 and kept track of the number of rolls she needed. The results of her 100 trials are shown in the histogram at the right. Using the data in the histogram, calculate the mean number of rolls needed to get a sum of 3, 4, or 5.



c. Using Keisha's data, estimate the probability that a player would have to roll more than five times before getting a 3, 4, or 5.

d. Using probability formulas and rules, rather than Keisha's data, find the probability that a player would first get a 3, 4, or 5 on one of the first three rolls of the dice.

LESSON 4

Statistical Reasoning

In the first three lessons of this unit, you explored how mathematical statements can be proved using algebraic and geometric reasoning. The reasoning was deductive—it involved showing that the conclusion followed logically from definitions, accepted postulates or properties, and theorems. If the reasoning was correct, you could be certain about the truth of the conclusion. However, problems in life that involve data also involve variability, randomness, or incomplete information. This means that you can never be completely certain that you have come to the correct conclusion. In such situations, statistical reasoning can help you come to a reasonable conclusion, backed by convincing evidence.

Advances in fields such as science, education, and medicine depend on determining whether a change in treatment causes better results.

Is a particular drug more effective against acne than doing nothing?

Does a sprain heal faster if the part is exercised or if it is rested?

Does listening to Mozart make people smarter?

You can get convincing evidence about such questions only by carrying out properly designed experiments.

With your class, perform an experiment to decide whether students can, on average, stack more pennies with their dominant hand (the hand with which they write) or with their nondominant hand. First, agree on the rules. Can you touch a penny again after you have stacked it? Can your elbow rest on the table? Then, randomly divide your class into two groups of about equal size. The students in one group will stack pennies using only their dominant hand. The students in the other group will use their nondominant hand. Each student will count the number of pennies he or she stacks before a penny falls.

Think About This Situation

Think about the design of this penny-stacking experiment and how you would interpret the results.

a) Why is it important to agree on rules (a *protocol*) for how you must stack the pennies?

b) Why is it important to divide your class into the two groups at random?

c) Complete this experiment and then organize your data using plots and summary statistics. (Save the data, as you will need them in Investigation 2.)

d) What can you conclude? Have you *proved* that, for students your age, one hand tends to be better than the other in stacking pennies? Why or why not?

In this lesson, you will learn how to design a good experiment and how to use statistical reasoning to decide whether one treatment (using the dominant hand) causes a better result (number of pennies stacked) than another treatment (using the nondominant hand). You will also explore reasoning from a sample to a population as in the case of predicting the results of an election from polling a sample of voters.

Investigation 1 Design of Experiments

Statistical reasoning involves these steps.

- Formulate a question that can be answered with data.
- Collect data.
- Display and summarize the data.
- Interpret the results and make generalizations.

In earlier statistics units of *Core-Plus Mathematics*, you concentrated mostly on the third and fourth steps, plotting data that had already been collected, computing summary statistics, and interpreting the results. In this investigation, you will focus on the second step. As you work on the problems in this investigation, look for answers to this question:

How can you design an experiment that provides convincing evidence that one treatment causes a different response than another treatment?

1. A common science experiment attempts to determine if mung bean seeds that are given a gentle zap in a microwave oven are more likely to sprout than mung bean seeds that are not given a zap.

Mung beans that were zapped in a microwave oven

a. In such an experiment, what are the **treatments** (conditions you want to compare)? What is the **response variable** (outcome you are measuring)?

b. For his experiment, Carlos zapped 10 mung bean seeds and 8 sprouted. Explain why Carlos should not conclude that mung bean seeds zapped in a microwave are more likely to sprout than if they had not been zapped.

c. For her experiment, Mia took 20 mung bean seeds, picked out 10 that looked healthy and zapped them. Of the 10 that were zapped, 8 sprouted. Of the 10 that were not zapped, 3 sprouted. Explain why Mia should not conclude that mung bean seeds zapped in a microwave are more likely to sprout than if they had not been zapped.

d. For her experiment, Julia took 4 mung bean seeds, selected 2 at random to be zapped, and zapped those 2. Both seeds that were zapped sprouted. The 2 seeds that were not zapped did not sprout. Explain why Julia should not conclude that mung bean seeds zapped in a microwave are more likely to sprout than if they had not been zapped.

e. Design an experiment to determine if mung bean seeds are more likely to sprout if they are zapped in a microwave.

2 In a typical experiment, two or more treatments are randomly assigned to an available group of people (or animals, plants, or objects) called **subjects**. The purpose of an experiment is to establish cause and effect. Does one treatment cause a different response than the other treatment? A well-designed experiment must have three characteristics.

- *Random assignment*: Treatments are assigned randomly to the subjects.
- *Sufficient number of subjects*: Subjects will vary in their responses, even when they are treated alike. If there are not enough subjects, this variability within each treatment may obscure any difference between the effects of the treatments. Deciding how many subjects are sufficient is one of the more difficult tasks that statisticians do.
- *Comparison group or control group*: Either the group that gets the treatment is compared to a group that gets no treatment (a **control group**) or two groups that get different treatments are compared.

a. Which characteristic(s) of a well-designed experiment was (were) missing in Problem 1 in the mung bean seed study of:

i. Carlos? **ii.** Mia? **iii.** Julia?

b. Which characteristics of a well-designed experiment, if any, were missing from your penny-stacking experiment?

c. What can go wrong if treatments are not assigned randomly to the subjects?

3 In 1954, a huge medical experiment was carried out to test whether a newly developed vaccine by Jonas Salk was effective in preventing polio. Over 400,000 children participated in the portion of the study described here. Children were randomly assigned to one of two treatments. One group received a *placebo* (an injection that looked—and felt!—like a regular immunization but contained only salt water). The other group received an injection of the Salk vaccine. (Source: Paul Meier, "The Biggest Public Health Experiment Ever," in *Statistics: A Guide to the Unknown*, 3rd ed. Edited by Judith Tanur, *et al*. Pacific Grove, CA: Wadsworth and Brooks/Cole Advanced Books and Software, 1989, pp. 3–14.)

Jonas Salk examining vials of polio vaccine

a. What are the treatments in the Salk experiment? What is the response variable?

b. Did the test of the Salk vaccine have the three characteristics of a well-designed experiment?

4 Many difficulties in testing the Salk vaccine had been anticipated. Which of the three characteristics of a well-designed experiment helped overcome each difficulty described below? Explain.

a. The incidence of polio was very low, even without immunization.

b. The vaccine was not expected to be 100% effective.

c. One possible approach would have been to immunize all children in the study and compare the incidence of polio to that of children the same age the previous year. However, the incidence of polio varied widely from year to year.

d. One possible experimental design would have been to let parents decide whether their child was vaccinated and compare the rates of polio of the vaccinated and unvaccinated children. In the United States, polio was primarily a disease of children from middle- and upper-income families and so those children's parents were especially anxious to get them vaccinated.

5 Many studies have shown that people tend to do better when they are given special attention or when they believe they are getting competent medical care. This is called the **placebo effect**. Even people with post-surgical pain report less discomfort if they are given a pill that is actually a placebo (a pill containing no medicine) but which they believe contains a painkiller.

One way to control for the placebo effect is to make the experiment **subject blind**, the person receiving the treatment does not know which treatment he or she is getting. That is, subjects in both treatment groups appear to be treated exactly the same way.

In an **evaluator-blind** experiment, the person who evaluates how well the treatment works does not know which treatment the subject received. If an experiment is both subject blind and evaluator blind, it is called **double blind**.

a. The Salk experiment was double blind. One reason this was necessary was because the diagnosis of polio is not clear-cut. Cases that cause paralysis are obvious, but they are the exception. Sometimes polio looks like a bad cold, and so professional judgment is needed. How might a doctor's knowledge of whether or not a child had been immunized affect his or her diagnosis? How might this lead to the wrong conclusion about how well the vaccine works?

b. Could you make the penny-stacking experiment in the Think About This Situation subject blind? Evaluator blind? Double blind? Explain.

6 A **lurking variable** helps to explain the association between the treatments and the response but is not the explanation that the study was designed to test. Treatments are assigned randomly to subjects to equalize the effects of possible lurking variables among the treatment groups as much as possible. Analyze each of the following reports of studies with particular attention to possible lurking variables.

a. Researchers from the Minnesota Antibiotic Resistance Collaborative reported an attempt to deal with the problem that bacteria are becoming resistant to antibiotics. One reason for increasing resistance is that some people want antibiotics when they have a cold, even though cold viruses do not respond to antibiotics.

Five medical clinics distributed colorful kits containing Tylenol®, decongestant, cough syrup, lozenges, powdered chicken soup, and a tea bag to patients with cold symptoms. At five other medical clinics, patients with similar symptoms were not given these kits. Patients with colds who visited clinics that made the kits available were less likely to fill prescriptions for antibiotics than patients with colds who visited clinics where the kits were not available. This study involved nearly 11,000 patients. (Source: sciencedaily.com/releases/2004/02/040229231510.htm; *The Los Angeles Times*, March 8, 2004.)

i. What are the treatments in the study? What is the response variable?

ii. Why is this not a well-designed experiment? How could you improve it?

iii. What lurking variable might account for the difference in response?

b. Researchers supplied 238 New York City households with hand-washing soaps, laundry detergents, and kitchen cleansers. Half of the households, selected at random, were given antibacterial products, and the other half received products that were identically packaged but without the antibacterial ingredient.

The participants were asked weekly about any disease in the household. The researchers found no difference in frequency of infectious disease symptoms over one year. (Source: Elaine L. Larson *et al*. "Effect of Antibacterial Home Cleaning and Handwashing Products on Infectious Disease Symptoms," *Annals of Internal Medicine*, Vol. 140, No. 5, March 2, 2004.)

i. Does this study have the three characteristics of a well-designed experiment?

ii. This study was double blind. Explain how that must have been done.

iii. Suppose that instead of assigning the treatments at random to the households, the researchers simply compare the frequency of infectious disease symptoms over a year in households that use antibacterial products and those that do not. Describe lurking variables that might invalidate the conclusion of the study.

c. A December 2004 article on washingtonpost.com entitled "In AP-vs.-IB Debate, A Win for the Students" reports on a study by the National Center for Educational Accountability that shows that "even students who fail AP examinations in high school are twice as likely to graduate from college in five years as students who never try AP." This study followed 78,079 students in Texas.

- **i.** What are the treatments? What is the response variable?
- **ii.** Do you think that the conclusion came from a well-designed experiment?
- **iii.** What lurking variables could account for the difference in responses for the two groups?
- **iv.** Can you design an experiment to establish that taking an AP course, even if you fail the exam, means you are more likely to graduate from college in five years?

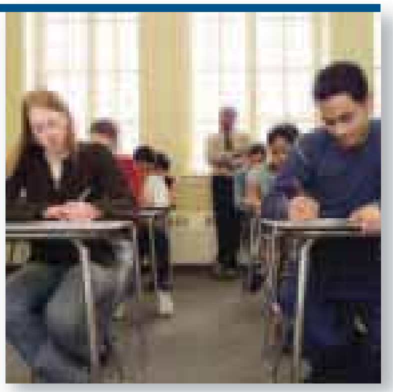
Summarize the Mathematics

In this investigation, you examined characteristics of well-designed experiments.

- **a** What are the three characteristics of a well-designed experiment? Why is each necessary?
- **b** Why are subject blinding and evaluator blinding desirable in an experiment?
- **c** What is the placebo effect? How can you account for it when designing an experiment?

Be prepared to share your ideas with the class.

Check Your Understanding



To find out if students do better on exams where the easier problems come first, a teacher wrote two versions of an exam. One had the more difficult problems first, and a second contained the same problems only with the easier ones first. He gave the first version to his first-period class and the second version to his second-period class. He will compare the scores on the first version with the scores on the second version.

a. What are the treatments? What is the response variable?

b. Does this study have the three characteristics of a well-designed experiment?

c. Is this study subject blind? Is it evaluator blind?

d. Name at least one lurking variable for this study.

e. Describe how the teacher could improve the design of his study.

Investigation 2 By Chance or from Cause?

In the previous investigation, you learned the importance of randomly assigning treatments to the subjects in an experiment. With randomization, you trust that any initial differences among the subjects get spread out fairly evenly between the two treatment groups. Consequently, you feel justified in concluding that any large difference in response between the two groups is due to the effect of the treatments. In this investigation, you will learn a technique for making *inferences* (drawing valid conclusions) from an experiment.

As you work on the problems in this investigation, look for answers to this question:

How can you decide whether the difference in the mean responses for the two groups in your experiment happened just by chance or was caused by the treatments?

1 To illustrate what happens when treatments are equally effective, your class will perform a simple but well-designed experiment to determine whether a calculator helps students perform better, on average, on a test about factoring numbers into primes.

- Each student should write his or her name on a small card.
- Divide your class at random into two groups of about equal size by having one person shuffle the name cards and deal them alternatively into two piles. One group will get the "calculator" treatment and the other group will get the "pencil-and-paper" treatment.
- Using the treatment assigned to you, answer the questions on the test provided by your teacher. If you do not know an answer for sure, make the best guess you can so that you have an answer for all eight questions.
- Grade your test from the answers provided by your teacher. Report the number of questions that you got correct as your response.

a. Was a calculator any help on this test? Should the treatment you received make any difference in your response?

b. Make a list of the responses for each treatment. Then, for each treatment, prepare a plot of the number correct and calculate appropriate summary statistics.

c. To the nearest hundredth, compute the difference: *mean of calculator group − mean of pencil-and-paper group*. Is there a difference in the mean response from the two treatment groups?

d. Why did you get a difference that is not zero even though which treatment you received should not matter?

2 Now, explore what the size of the difference in the mean responses in Problem 1 Part c would have been if the randomization had placed different students in the two treatment groups, but each student's response was exactly the same.

a. On each name card, record the number of correct responses for that student.

i. Divide your class at random once again by shuffling the name cards and dealing them into two piles representing the calculator treatment and the pencil-and-paper treatment. To the nearest hundredth, what is the difference: *mean of calculator group* − *mean of pencil-and-paper group*?

ii. Construct an approximate distribution of the possible differences of means by repeating the above process. Do this, sharing the work, until you have generated a total of at least 50 differences.

b. Where is your distribution centered? Why does this make sense?

c. Is the difference in the means from the original experiment (Problem 1 Part c) so extreme that it falls in the outer 5% of the distribution? In other words, would the difference be a rare event if it had been generated by chance alone?

d. If so, what can you conclude? If not, what can you conclude?

3 Researchers at the Smell & Taste Foundation were interested in the following question.

Can pleasant aromas improve ability to complete a task?

They randomly assigned volunteers to wear an unscented mask or to wear a floral-scented mask. The subjects then completed two pencil-and-paper mazes. The time to complete the two mazes was recorded. Data were recorded separately for smokers and nonsmokers as smoking affects the sense of smell. Results for the 13 nonsmokers are given in the table below.

Unscented Mask (in seconds)	Scented Mask (in seconds)
38.4	38.0
72.5	35.0
82.8	60.1
50.4	44.3
32.8	47.9
40.9	46.2
56.3	

Source: Hirsch, A. R., and Johnston, L. H. "Odors and Learning," Smell & Taste Treatment and Research Foundation, Chicago. DASL library, lib.stat.cmu.edu/DASL/Datafiles/Scents.html

a. Does this study have the three characteristics of a well-designed experiment?

b. Why did some subjects have to wear an unscented mask?

c. Compute summary statistics and make plots to help you decide whether one type of mask results in shorter times to complete the mazes.

d. From your summaries, you found that people who wore the unscented mask took an average of 8.19 seconds longer to complete the mazes than people who wore the scented mask. Do you think that the scented mask causes a shorter mean time to complete the mazes or do you think that the difference of 8.19 seconds is no more extreme than you would expect just by chance?

4 Assume that the type of mask in Problem 3 makes absolutely no difference in how long it takes a person to complete the mazes.

a. Why would you expect there to be a nonzero difference anyway in the mean times for the two treatments?

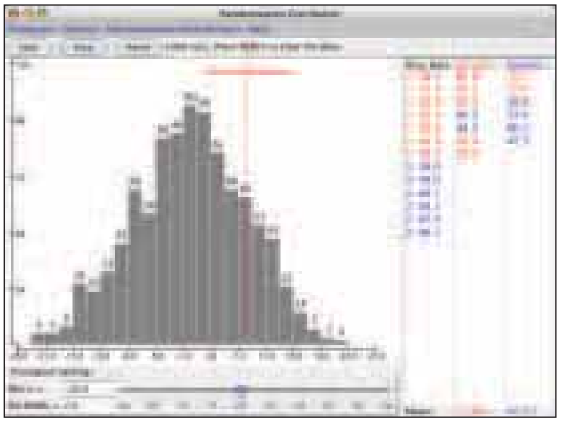
b. How long should it take the people who wore the unscented masks to complete the mazes if they had worn scented masks instead?

c. Now suppose that you are the researcher beginning this experiment and need to pick 7 of the 13 nonsmokers to wear the unscented mask.

- Write, on identical slips of paper, the 13 maze-completing times given in the table on page 82.
- Mix them up well. Draw 7 of them to represent the 7 people who will wear the unscented mask. Compute their mean time. Compute the mean time of the remaining 6 people.
- Subtract: *unscented mean − scented mean*.

Compare your result with that of others.

d. A faster way to construct an approximate distribution from many randomizations is to use statistical software to randomly select groups. The "Randomization Distribution" feature of *CPMP-Tools* created this approximate distribution of possible differences (*unscented mean* − *scented mean*).



The display shows 1,000 runs of randomly assigning the 13 times required to complete the mazes to either the unscented mask treatment (7 times) or the scented mask treatment (6 times). Out of these 1,000 differences, about how many times is the difference at least as extreme as 8.19 seconds? If the scent made absolutely no difference, estimate the probability that you would get a difference as extreme as 8.19 seconds just by random chance.

e. Which of the following is the best conclusion?

- The difference of 8.19 from the actual experiment is extreme, so you should abandon your supposition that the scent made no difference.
- It is quite plausible that the scent does not affect the time to complete the maze. In other words, a difference of 8.19 seconds would not be unusual if the scent made no difference and you randomly divide the subjects into two groups.

5 Look back at the results of your penny-stacking experiment in the Think About This Situation on page 75.

a. Compute the difference for your class: *mean number of pennies stacked by those using their dominant hand* − *mean for those using their nondominant hand*. Suppose that the hand people use makes absolutely no difference in how many pennies they can stack. Why, then, would there almost always be a nonzero difference in every class that does this experiment?

b. If which hand you used made absolutely no difference, how many pennies would you have been able to stack if you had used your other hand?

c. Use the "Randomization Distribution" feature of *CPMP-Tools* to create an approximate distribution of the possible differences (*dominant hand mean* − *nondominant hand mean*). Run at least 500 random assignments, continuing until the shape of the distribution stabilizes.

d. How many times did you get a simulated difference as extreme as what you got in your experiment? If which hand you use makes no difference, estimate the probability that you would get a difference as extreme as what you got in your experiment just by random chance.

e. Should you conclude that the hand you used made a difference?

The reasoning that you followed in Problems 1–5 to decide whether the results of an experiment provide convincing evidence that different treatments cause a different mean response is called a **randomization test** (or, sometimes, **permutation test**). The steps below summarize this reasoning.

Step 1. Assume that which treatment each subject gets makes absolutely no difference in his or her response. In other words, assume the subjects in the experiment would give the same response no matter which treatment they receive. In the next two steps, you will see if this assumption is plausible.

Step 2. Simulate the experiment.

- Write the name of each subject along with his or her response on a card.
- Randomly divide the cards into two treatment groups.
- Compute the mean for each treatment group.
- Find the difference of these means.
- Repeat this many times until you can see the shape of the distribution of differences.

Step 3. Locate the difference from the actual experiment on the distribution you generated in Step 2.

Step 4. If the difference from the actual experiment is in the outer 5% of the distribution, conclude that the results are **statistically significant**. That is, you have evidence that the treatments caused the difference in the mean response. If the difference is not in the outer 5% of the distribution, conclude that your original assumption was plausible. The difference can be reasonably attributed solely to the particular random assignment of treatments to subjects.

6 Chrysanthemums with long stems are likely to have smaller flowers than chrysanthemums with shorter stems. An experiment was conducted at the University of Florida to compare growth inhibitors designed to reduce the length of the stems, and so, increase the size of the flowers. Growth inhibitor A was given to 10 randomly selected plants. Growth inhibitor B was given to the remaining 10 plants. The plants were grown under nearly identical conditions, except for the growth inhibitor used. The table below gives the amount of growth during the subsequent 10 weeks.

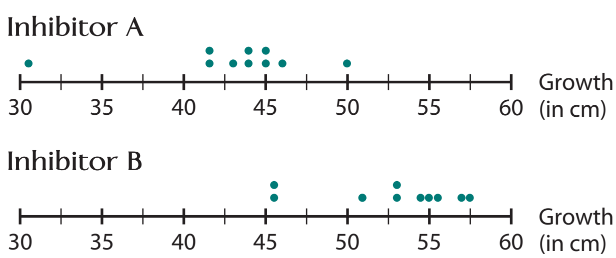
Growth by Plants Given A (in cm)	Growth by Plants Given B (in cm)
46	51
41.5	55
45	57
44	57.5
41.5	53
50	45.5
45	53
43	54.5
44	55.5
30.5	45.5

Source: Ann E. Watkins, Richard L. Scheaffer, and George W. Cobb, *Statistics in Action*, 2nd Ed. Key Curriculum Press, 2008, p. 802.

a. Does this experiment have the three characteristics of a well-designed experiment?

b. Examine the following summary statistics and plots. Which growth inhibitor treatment appears to be better?

Treatment	Mean	Standard Deviation	Number of Plants
A	43.05	5.04	10
B	52.75	4.28	10



c. Describe how to use a randomization test to decide whether, on average, one growth inhibitor works better than the other.

d. Use the "Randomization Distribution" feature of *CPMP-Tools* to create an approximate distribution of possible differences (*growth inhibitor A mean* − *growth inhibitor B mean*). Run at least 500 random assignments.

e. If the type of growth inhibitor makes no difference, what is your estimate of the probability of getting a difference at least as extreme as the difference from the actual experiment?

f. What is your conclusion? Is the difference statistically significant?

Summarize the Mathematics

In this investigation, you explored the randomization test. This test is one method of determining whether a difference between two treatment groups can be reasonably attributed to the random assignment of treatments to subjects or whether you should believe that the treatments caused the difference.

a Explain why this statement is true: Even if the response for each subject would be the same no matter which treatment he or she receives, there is almost always a nonzero difference in the means of the actual responses from the two treatments.

b What does it mean if the results of an experiment are called "statistically significant"?

c Explain the reasoning behind the steps of a randomization test.

d Explain how this statement applies to the reasoning in this unit: Statistical reasoning is different from mathematical proof because in statistics, you can never say you are certain.

Be prepared to share your responses and thinking with the class.

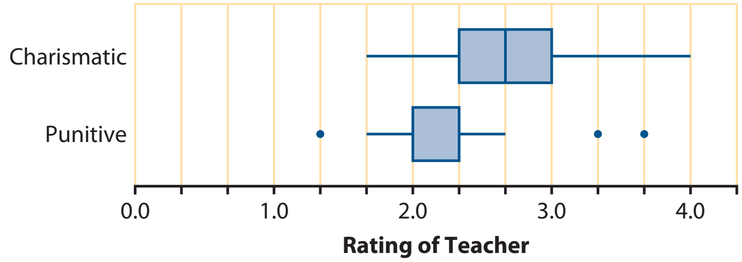
Check Your Understanding

Forty-nine volunteer college students were randomly assigned to two treatments. Twenty-five students were told that they would view a video of a teacher who other students thought was "charismatic": lively, stimulating, and encouraging. The remaining twenty-four students were told that the instructor they would view was thought to be "punitive": not helpful, not interested in students, and a hard grader. Then all students watched the same twenty-minute lecture given by the same instructor. Following the lecture, subjects rated the lecturer. The students' summary ratings are given below. Higher ratings are better.

Charismatic: $1\frac{2}{3}$, 3, $1\frac{2}{3}$, $2\frac{1}{3}$, 4, $2\frac{1}{3}$, 2, $2\frac{2}{3}$, $2\frac{2}{3}$, $2\frac{1}{3}$, $3\frac{1}{3}$, $2\frac{1}{3}$, $2\frac{1}{3}$, $2\frac{2}{3}$, 3, $2\frac{2}{3}$, 3, 2, $2\frac{1}{3}$, $2\frac{2}{3}$, 3, $3\frac{1}{3}$, 3, $2\frac{2}{3}$, $2\frac{1}{3}$

Punitive: $2\frac{2}{3}$, 2, 2, $1\frac{1}{3}$, $1\frac{2}{3}$, $2\frac{1}{3}$, $2\frac{2}{3}$, 2, $1\frac{2}{3}$, $1\frac{1}{3}$, $2\frac{1}{3}$, 2, $2\frac{1}{3}$, $2\frac{1}{3}$, $2\frac{1}{3}$, $2\frac{1}{3}$, $1\frac{2}{3}$, $3\frac{2}{3}$, $2\frac{1}{3}$, $2\frac{2}{3}$, 2, $2\frac{1}{3}$, $2\frac{1}{3}$, $3\frac{1}{3}$

Treatment	Mean	Standard Deviation	Number of Students
Charismatic	2.61	0.53	25
Punitive	2.24	0.54	24



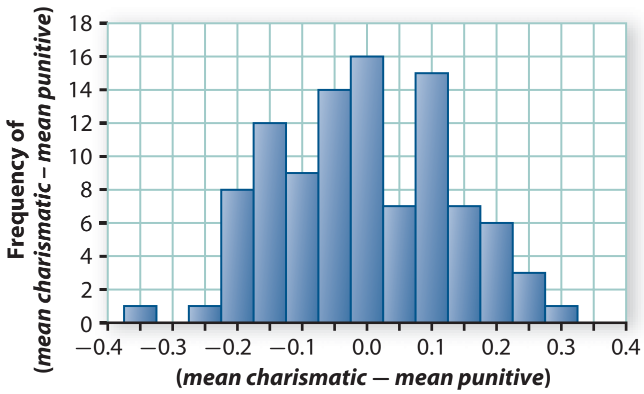
Source: www.ruf.rice.edu/%7Elane/case_studies/instructor_reputation/index.html (Their source: Towler, Annette., & Dipboye, R. L. (1998). The effect of instructor reputation and need for cognition on student behavior—poster presented at American Psychological Society conference, May 1998.)

a. What are the two treatments?

b. From the box plots and summary statistics, does it look like the two treatments cause different responses? Explain.

c. Describe how to perform one run for a randomization test to decide whether the two different treatments result in different mean ratings.

d. The randomization distribution below shows *mean charismatic* − *mean punitive* for 100 runs. Use the histogram to estimate the probability of getting a difference as extreme as that from the actual experiment if the treatment makes no difference.



e. What is your conclusion? Is the difference statistically significant?

Investigation 3 Statistical Studies

Experiments are one of the three major types of statistical studies. The other two are *sample surveys* and *observational studies*. As you work on the problems in this investigation, look for answers to these questions:

What are the differences between sample surveys, experiments, and observational studies?

What kind of conclusions can be made from each?

The three main types of statistical studies are described below.

- ***sample survey or poll***: You observe a random sample in order to estimate a characteristic of the larger population from which the sample was taken. Getting a **random sample of size *n*** is equivalent to writing the name of every member of the population on a card, mixing the cards well, and drawing n cards.
- ***experiment***: You randomly assign two (or more) treatments to the available subjects in order to see which treatment is the most effective.
- ***observational study***: The conditions you want to compare come already built into the subjects that are observed. Typically, no randomization is involved.

1 Suppose you want to investigate the effects of exercise on the blood pressure of students in your school. You have thought about three different study designs. Classify each design as a sample survey, an experiment, or an observational study.

Study 1: You ask for volunteers from the students in your school and get 30 students willing to participate in your study. You randomly divide them into two groups of 15 students. You ask one group not to exercise at all for the next week, and you ask the other group to do at least 30 minutes of exercise each day. At the end of the week, you find that everyone complied with your instructions. You then take each student's blood pressure. You find that the mean blood pressure of the students who exercised is lower than the mean blood pressure of the students who did not exercise.

Study 2: You get a list of all students in your school and use a random digit table to select 30 of them for your study. You take these students' blood pressure and then have them fill out a questionnaire about how much exercise they get. You divide them into those who exercise a lot and those who exercise less. You find that the mean blood pressure of the students who exercise more is lower than the mean blood pressure of the students who exercise less.

Study 3: You discover that the nurse in the health office at your school has taken the blood pressure of 157 students who have visited the health office over the past year for a variety of reasons. In some cases, they felt sick; and in other cases, they had to turn in routine paperwork. You get the names of these students and have them fill out a questionnaire about how much exercise they get. You find that the mean blood pressure of the students who exercise more is lower than the mean blood pressure of the students who exercise less.

2. In each study in Problem 1, there was an association between amount of exercise and blood pressure. Assume that in each case the difference in mean blood pressure was statistically significant. Answer the following questions for each study in Problem 1.
   a. Is it reasonable to conclude that it was the exercise that caused the lower blood pressure? Explain your thinking.
   b. Can you generalize the results of this study to all of the students in your school? Explain your thinking.
   c. Exactly what can you conclude from this study?

3. Refer to Problem 6 Part c on page 80 about students who take AP examinations in high school.
   a. What type of study is this?
   b. State the conclusion that can be drawn.

4. Every four years, the Gallup organization tries to predict the winner of the U.S. presidential election. They do this by first creating a list of all possible household phone numbers in the United States. They then phone several thousand households using random digit dialing, calling back if no one answers. An adult is selected at random from each household and interviewed about whether he or she intends to vote and for whom. (Source: Frank Newport, Lydia Saad, David Moore "How are polls conducted?" *Where America Stands*, Wiley, 1997, media.gallup.com/PDF/FAQ/HowArePolls.pdf)
   a. What type of study is this?
   b. Explain why households cannot be selected from phone books.
   c. Are all adults in the United States equally likely to be in the sample? Explain.

Summarize the Mathematics

In this investigation, you examined the three main types of statistical studies.

a. What is a random sample?

b. How is randomization used in a sample survey? In an experiment? In an observational study?

c. What kind of conclusion can you draw from a sample survey? From an experiment? From an observational study?

Be prepared to share your responses with the class.

Check Your Understanding

The British Doctors Study was one of the earliest studies to establish a link between smoking and lung cancer. In 1951, all male doctors in the United Kingdom were contacted. About two-thirds, or 34,439 doctors, agreed to participate. Eventually, researchers found that the doctors who smoked were more likely to get lung cancer than doctors who did not smoke. The difference was statistically significant. (Source: Richard Doll, *et al*. "Mortality in Relation to Smoking: 50 Years' Observations on Male British Doctors," *British Medical Journal*, Vol. 22, June 2004, www.bmj.com/cgi/reprint/bmj.38142.554479.AEv1)

a. What type of study is this?

b. Can you conclude from this study that smoking causes lung cancer? Explain your thinking.

c. Can you generalize the results of this study to some larger population? Explain your thinking.

d. Describe exactly what you can conclude from this study.

On Your Own

Applications

1 Suppose that the manufacturer of a cough medicine wants to conduct a randomized, double-blind experiment to determine if adding a new ingredient results in a reduction in the mean number of coughs per hour. Fifty adult volunteers with persistent coughs are available. Describe how the manufacturer should conduct this experiment.

2 The table below gives the overall driver death rate for various sizes of four-door cars. There is a strong association between the weight of a car and driver death rate. However, there are many advantages to lighter cars. They tend to be easier to park, to get better gas mileage, to be less polluting, to have cheaper insurance, and to be less expensive to buy. So, it is worth carefully considering whether they are really less safe.

Weight of Car (in lbs)	Overall Driver Death Rate (per million registered vehicle years)
2,500 or less	115
2,501–3,000	102
3,001–3,500	84
3,501–4,000	56
4,001–4,500	47

Source: *The Risk Of Dying In One Vehicle Versus Another: Driver Death Rates By Make And Model,* Insurance Institute for Highway Safety, Status Report special issue: March 19, 2005.

a. Describe the association between the weight of a car and driver death rate.

b. Suppose that 4,000,000 of a certain model of car are registered, and there were 467 driver deaths over a three-year period. What would be the overall driver death rate per million registered vehicle years for that particular model of car?

c. Name a possible lurking variable. Describe how it could account for the association between the weight of a car and driver death rate.

d. How could you design an experiment to provide convincing evidence that lighter cars cause a larger overall driver death rate than heavier cars? Would this be ethical?

3 Psychrotrophic bacteria cause meat to spoil. Six beef steaks were randomly assigned to be packaged using commercial plastic wrap or to be vacuum packaged. The following table gives the logarithm (log) of the number of psychrotrophic bacteria per square centimeter on the meat after nine days of storage at controlled temperature.

Commercial Plastic Wrap log (count/cm^2)	Vacuum Packaged log (count/cm^2)
7.66	5.26
6.98	5.44
7.80	5.80

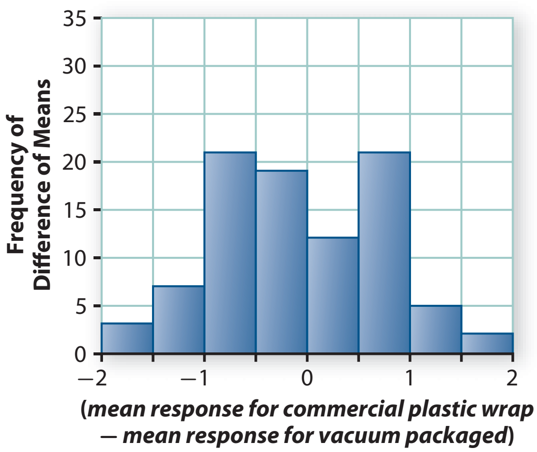
Source: Robert O. Kuehl, *Statistical Principles of Research Design and Analysis*, Duxbury Press, Belmont, CA, 1994, p. 31. Original source: B. Nichols, *Comparison of Grain-Fed and Grass-Fed Beef for Quality Changes When Packaged in Various Gas Atmospheres and Vacuum*, M.S. thesis, Department of Animal Science, University of Arizona, 1980.

a. How many bacteria per square centimeter were on the steak with a log of 7.66?

b. Does this study have the three characteristics of a well-designed experiment? What else would you like to know about how it was conducted?

c. What is the difference *mean (log) response for commercial plastic wrap − mean (log) response for vacuum packaged*?

d. Describe how to conduct a randomization test to decide whether the different packaging causes different numbers of bacteria. Perform 10 runs and add them to a copy of the randomization distribution below, which shows the results of 90 runs.



e. Use the randomization distribution to estimate the probability that random assignment alone will give you a difference that is at least as extreme as that from the real experiment. Do you have evidence from this experiment that one type of packaging is better than the other?

4 For a science project, Brian wanted to determine whether eleventh-graders did better when they took a math test in silence or when Mozart was being played. Twenty-six students were randomly divided into the two treatment groups. Part of Brian's results are in the table below.

Mozart (percentage correct)	Silence (percentage correct)
65	44
80	70
72	68
68	58
38	58
58	47
45	54
42	44
58	61
81	61
40	9
41	52
27	30
Mean = 55	**Mean ≈ 50.46**

a. What is the difference *mean response for Mozart group* − *mean response for silence group*?

b. Describe how to conduct a randomization test to decide whether the different treatments cause different performance on the math test.

c. Use the "Randomization Distribution" feature of your data analysis software to create an approximate distribution of the possible differences (*mean response for Mozart group* − *mean response for silence group*). Run at least 500 random assignments.

d. Use the randomization distribution to estimate the probability that you could get a difference, just by the random assignment, that is as extreme as that from the real experiment.

e. Brian concluded that students who listen to Mozart during a test tend to do better. Do you agree with this conclusion or do you think the difference can reasonably be attributed to the random assignment alone?

5 You conduct a test to see if an inexperienced person gets a bigger geyser if he or she drops Mentos® candy into a liter bottle of cola by hand or by using a paper funnel. You find ten friends who have never done this demonstration and are willing to participate. You write "by hand" on five slips of paper and "funnel" on five slips of paper. Each person draws a slip and is shown how to use that method. Then, each person drops Mentos® into his or her bottle of cola using the method assigned. The maximum height of each geyser is measured.

You find that the difference between the mean heights of the geysers produced by your friends who used their hand and your friends who used a paper funnel is not statistically significant.

a. What type of study is this?

b. What are the treatments? What are the subjects?

c. Can you generalize the results of this study to some larger population? Explain your thinking.

d. Describe exactly what you can conclude from this study.

6 Researchers wanted to determine whether social class is related to smoking behavior. They conducted telephone interviews with 1,308 Massachusetts adolescents aged 12 to 17, selected by dialing at random. They found a statistically significant association between whether the adolescent smoked or not and the household income. Adolescents from households with less income were more likely to smoke, and this was true across all ages, for both sexes, for all races, and for all amounts of disposable income the adolescent had. (Source: Elpidoforos S. Soteriades and Joseph R. DiFranza. "Parent's Socioeconomic Status, Adolescents' Disposable Income, and Adolescents' Smoking Status in Massachusetts," *Journal of Public Health,* Vol. 93, July 2003, pp. 1155–1160, www.pubmedcentral.nih.gov/articlerender.fcgi?artid=1447926)

a. What type of study is this?

b. Can you conclude from this study that smoking is caused by an adolescent's social class? Can you think of a lurking variable that might be responsible for both?

c. Can you generalize the results of this study to some larger population? Explain your thinking.

d. Describe exactly what you can conclude from this study.

Connections

7 In mathematics, if you have one counterexample, it disproves a conjecture. For example, in Lesson 1, you disproved the conjecture that all numbers of the form $n^2 - n + 41$ are prime, where n is a whole number 0, 1, 2, 3, … . To disprove it, all you had to do was find one value of n for which this conjecture is not true.

However, anecdotal evidence, looking at just one counterexample, is the *worst* kind of statistical reasoning. For example, you should not conclude that smoking does not cause lung cancer just because you know someone who smoked all his life and died at age 98 in a traffic accident. Even though not everyone who smokes gets lung cancer, explain in what sense it is valid to say that smoking causes lung cancer.

8 Look back at Applications Task 3. Because the sample sizes are so small, you can list all possible randomizations.

a. List all 20 of the possible selections of three of the six steaks to get the commercial plastic wrap and three to get vacuum packaging. Assuming that the type of packaging did not affect the response, compute the 20 possible differences *mean response for commercial plastic wrap − mean response for vacuum packaged*.

b. What is the probability that you get a difference just through a random assignment of steaks to packaging that is as extreme as that from the real experiment?

c. Do you have evidence from this experiment that one type of packaging is better than the other? Explain.

d. Is your conclusion consistent with that from the randomization test in Applications Task 3?

9 The reasoning of a randomization test in statistics is similar to a special form of indirect reasoning in mathematics called *proof by contradiction*. (See Extensions Tasks 25 and 26 on page 49.) To prove that a statement is not true using proof by contradiction, you follow these steps.

- Assume the statement is true.
- Show that this assumption leads to a contradiction.
- Conclude that the statement is not true.

It is a theorem (proven fact) that if a number p is a prime, then when you divide $2^p - 2$ by p, the remainder is 0.

a. Show that this theorem holds for the primes 5, 7, and 11.

b. Use this theorem and the three steps above to prove by contradiction that 21 is not a prime. Begin by assuming that 21 is a prime.

c. Explain how using the reasoning of a randomization test to conclude that two treatments cause different mean responses is similar to proof by contradiction.

10 In the Salk experiment, 82 of the 200,745 children who received the Salk vaccine were diagnosed with polio. Of the 201,229 children who received the placebo, 162 were diagnosed with polio.

a. What proportion of children who received the Salk vaccine were diagnosed with polio? What proportion who received the placebo were diagnosed with polio?

b. Do you think the result of this experiment is statistically significant or do you think the difference in the proportions is about the size that you would expect just by chance?

c. The number of children in the Salk experiment may seem excessive, but it was necessary. In the 1950s, before the Salk vaccine, the rate of polio in the U.S. was about 50 per 100,000 children. Suppose the experiment had "only" 4,000 children in the placebo injection group and 4,000 children in the Salk vaccine group. Also, suppose the vaccine is 50% effective; that is, it eliminates half of the cases of polio.

i. How many children in the placebo group would you expect to get polio?

ii. How many children in the Salk vaccine group would you expect to get polio?

iii. Does the difference now appear to be statistically significant?

d. As another part of the Salk experiment, in some schools, parents of second-grade children decided whether the child would be vaccinated or not. Among the 123,605 second-grade children whose parents did not give permission for them to receive the Salk vaccine, there were 66 cases of polio. In other schools, children were selected at random to receive the vaccine or the placebo from among those children whose parents gave permission for them to be in the experiment. In those schools, there were 162 cases of polio among the 201,229 children who received the placebo. How do you explain this result? (You may want to reread Problem 4 on page 78.)

Reflections

11 You may have seen a child fall and skin his knee, and his mother picks him up and says she will "kiss it and make it well." On what is the mother depending?

12 Mathematical arguments frequently use sentences beginning with the phrase *it follows that* … . They also often involve sentences connected with words like *because, therefore, so,* and *consequently*. Statistical arguments often involve phrases like *it is reasonable to conclude that* …, *there is strong evidence that* … , and *the data strongly suggest that* … . What is it about mathematical reasoning and statistical reasoning that explains the difference in choice of words and phrases?

13 How are the experiments as described in this lesson similar to and different from experiments you have conducted in your previous mathematical studies?

14 Why is it the case that an experiment can never *prove* without any doubt that two treatments cause different responses?

Extensions

15 Joseph Lister (1827–1912), surgeon at the Glasgow Royal Infirmary, was one of the first to believe in the theory of Louis Pasteur (1822–1895) that germs cause infection. In an early medical experiment, Lister disinfected the operating room with carbolic acid before 40 operations. He did not disinfect the operating room before another 35 operations. Of the 40 operations in which carbolic acid was used, 34 patients lived. Of the 35 operations in which carbolic acid was not used, 19 patients lived.

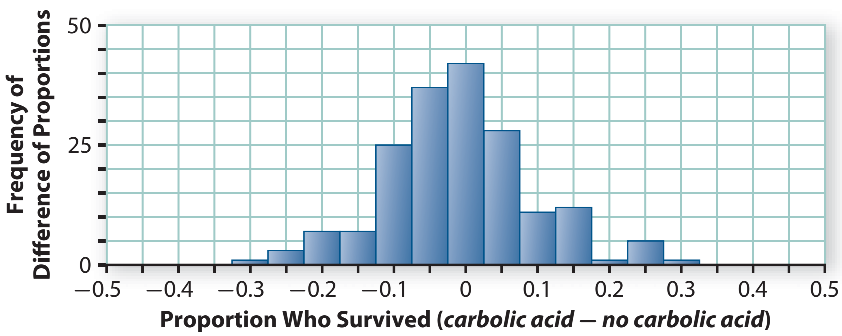
a. Why did Lister need to have one group of patients for whom he did not disinfect the operating room?

b. What is the difference in the proportion who lived if carbolic acid was used and the proportion who lived if carbolic acid was not used?

c. Does Lister's study provide convincing evidence that disinfecting operating rooms with carbolic acid results in fewer deaths than not disinfecting? Explain your thinking. Is there anything else you would like to know about how Lister conducted his experiment before you decide?

d. To begin a randomization test, you can let 0 represent a response of "died" and 1 represent a response of "lived." Describe how to finish the randomization test.

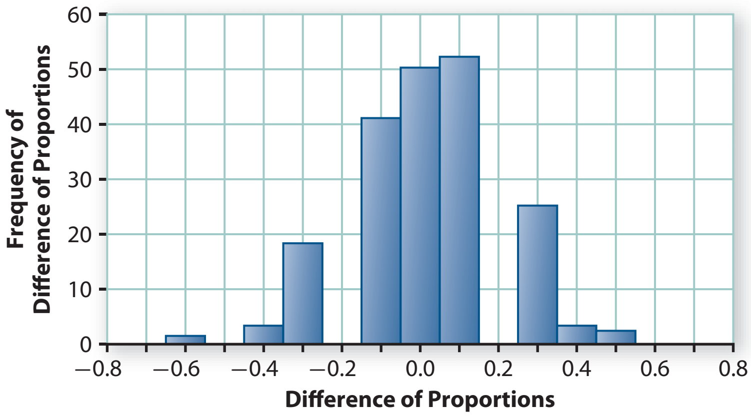
e. The randomization distribution below shows the results of 200 runs of the randomization. It records *proportion who survived when carbolic acid used – proportion who survived when carbolic acid not used*. Are the results of Lister's experiment statistically significant? Explain.



16 In a psychology experiment, a group of 17 female college students were told that they would be subjected to some painful electric shocks. A group of 13 female college students were told they would be subjected to some painless electric shocks. The subjects were given the choice of waiting with others or alone. (In fact, no one received any shocks.) Of the 17 students who were told they would get painful shocks, 12 chose to wait with others. Of the 13 students told they would get painless shocks, 4 chose to wait with others. (Source: Stanley Schachter. *The Psychology of Affiliation*, Stanford, CA: Stanford University Press, 1959, pp. 44–45.)

a. Describe how to use a randomization test to see if the difference in the proportions of students who choose to wait together is statistically significant. You can let 0 represent a response of waiting alone and 1 represent waiting with others.

b. Conduct 5 runs for your test and add them to a copy of the randomization distribution below, which shows 195 runs.



c. Is the difference in the proportions who choose to wait together statistically significant?

d. Why are there no differences between 0.15 and 0.25?

17 A market research experiment was designed to determine how much a subtle color change in white tennis shoes mattered to people who wore them. Twenty people who volunteered to try a new brand of tennis shoe were randomly assigned to get one of two colors of the same shoe. They wore the shoes for a month and then were told they could buy the pair of tennis shoes at a greatly reduced price or return them. Of the 10 people who got color A, 9 decided to buy them. Of the 10 people who got color B, 4 decided to buy them.

a. What kind of a study was this?

b. Describe the logic of a randomization test to determine if the difference is statistically significant and how to do such a test. You can let 0 represent a response of not buying the tennis shoes and 1 represent a response of buying the shoes.

c. Use the "Randomization Distribution" feature of your data analysis software to create an approximate distribution of the differences *mean number deciding to buy in group A − mean number deciding to buy in group B*. You will need to enter the data in a data sheet. Run at least 1,000 random assignments. What is your conclusion?

Review

18 Draw the graph of each equation.

a. $3x + 4y = 12$

b. $x - 5y = 7$

c. $x + y = 0$

d. $x = 5$

e. $y = 3$

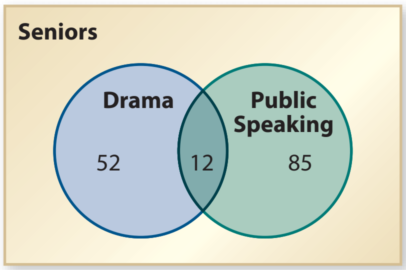
19 Triangle ABC has vertex matrix $\begin{bmatrix} 0 & 5 & 10 \\ 2 & 6 & 2 \end{bmatrix}$.

a. Is $\triangle ABC$ an isosceles triangle? Explain your reasoning.

b. The transformation $(x, y) \rightarrow (-x, y)$ is applied to $\triangle ABC$. Find the vertex matrix for the image $\triangle A'B'C'$ and describe this transformation.

c. Give a vertex matrix for a triangle that has an area 9 times the area of $\triangle ABC$.

20 The Venn diagram below indicates the number of seniors who have taken Drama and Public Speaking. There are 178 seniors.



a. What percentage of the seniors have taken both Drama and Public Speaking?

b. How many seniors have taken at least one of these courses?

c. How many seniors have not taken Drama and have not taken Public Speaking?

21 Solve each equation.

a. $x^2 + 10x + 22 = 6$

b. $2x - 3 = \frac{1}{x}$

22 Solve each inequality and graph the solution on a number line.

a. $3(x + 5) \geq 3$

b. $7x + 3 < 11x - 5$

c. $4x + 2(3 - 5x) > -12$

d. $10 - 2x \leq 6(x - 3) + 10$

23 Achmed kept track of the number of miles n he drove for several consecutive weeks and the amount of gas g in gallons he purchased each week. He found the linear regression equation relating these two variables to be $g = 0.035n + 0.4$.

a. Is the correlation for these two variables positive or negative? Explain your reasoning.

b. Explain the meaning of the slope of the regression equation in terms of these two variables.

c. During one week, Achmed drove 228 miles and bought 7.36 gallons of gas. Find the residual for that week.

24 Without using a calculator or computer graphing tool, sketch graphs of these quadratic functions. Then, for each, use the graph to determine the number of solutions for the related quadratic equation $f(x) = 0$.

a. $f(x) = x^2 - 4$

b. $f(x) = x^2$

c. $f(x) = x^2 + 4$

d. $f(x) = -x^2 - 5$

e. $f(x) = -x^2$

f. $f(x) = -x^2 + 4$

25 The height h in feet of a thrown basketball is a function of the time t in seconds since it was released. Suppose that Samuel shoots a free throw for which the height of the ball can be approximated by the function $h(t) = 6 + 40t - 16t^2$.

a. Find the value of $h(2)$ and explain what it tells you about the path of the basketball.

b. How high above the floor was the basketball when it was released?

c. For what values of t is $h(t) = 27$?

d. When was the basketball at its highest point and how high was it?

LESSON 5

Looking Back

In this unit, you examined and practiced reasoning principles and strategies that mathematicians and statisticians use to prove or justify their claims. In particular, you learned how to use definitions and assumed geometric facts to prove important properties of perpendicular and parallel lines. You learned how to use operations on algebraic expressions to prove patterns in number sequences, prove properties of real numbers, write equivalent expressions, and solve equations. You also learned how statisticians design experiments and use reasoning like the randomization test to determine whether one treatment causes a different outcome than another treatment.

The following tasks will help you to review and apply some of the key ideas involved in geometric, algebraic, and statistical reasoning.

1. **Reasoning about Shapes** In your earlier coursework, you saw that there were at least two different, but equivalent, definitions of a parallelogram. One of those definitions was: A **parallelogram** is a quadrilateral with two pairs of opposite sides parallel.

   a. Write two if-then statements that together mean the same thing as that definition.

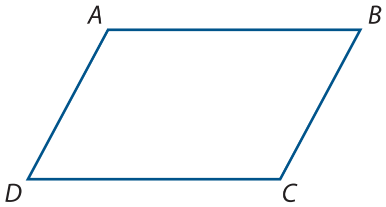
   b. Suppose you are told that in the case of quadrilateral *PQRS*, $\overline{PQ} \parallel \overline{SR}$ and $\overline{QR} \parallel \overline{PS}$. What can you conclude? Which of the two if-then statements in Part a was used in your reasoning?

   c. A definition of trapezoid was given in Lesson 1. A **trapezoid** is a quadrilateral with a pair of opposite sides parallel.

      i. Why is every parallelogram a trapezoid but not every trapezoid a parallelogram?

      ii. Use a Venn diagram to illustrate the relationship of parallelograms, trapezoids, and quadrilaterals.

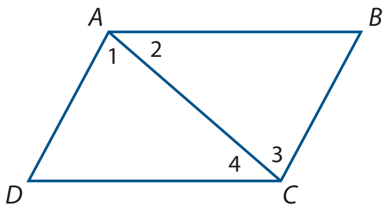
2 In a parallelogram *ABCD*, angles that share a common side, like $\angle A$ and $\angle B$, are called *consecutive angles*. Angles that do not share a common side, like $\angle A$ and $\angle C$, are called *opposite angles*.



a. Using the definition of a parallelogram and properties of parallel lines, prove that in $\square ABCD$, $\angle A$ and $\angle B$ are supplementary. Could you use a similar argument to prove other pairs of consecutive angles are supplementary? Explain your reasoning.

b. Use what you proved in Part a to help you prove that in $\square ABCD$, $m\angle A = m\angle C$. Does the other pair of opposite angles have equal measure? Explain your reasoning.

c. You have seen that there is often more than one correct way to prove a statement. Use the numbered angles in the diagram below to provide a different proof that $m\angle A = m\angle C$.



d. Write two statements summarizing what you proved in Parts a and b.

e. Give counterexamples to show the statements in Part d are not necessarily true for trapezoids.

3 **Reasoning about Patterns in Sequences** In Lesson 3, you discovered and proved an interesting pattern in the sequence of square numbers 1, 4, 9, 16, 25, 36, … . You found that the differences of successive terms in that sequence create the sequence of odd numbers 3, 5, 7, 9, 11, … . The *n*th number in that sequence of differences is given by the expression $2n + 1$.

a. Look again at the sequence of square numbers and study the pattern formed by calculating the differences that begins $9 - 1 = 8$, $16 - 4 = 12$, $25 - 9 = 16$, … .

i. What are the next several terms in this sequence of differences?

ii. What expression in the form "$an + b$" shows how to calculate the *n*th term in the sequence of differences?

iii. Use algebraic reasoning to prove that the expression $(n + 2)^2 - n^2$ is equivalent to the simpler expression you proposed in part ii.

b. Now consider the sequence of differences that begins $16 - 1 = 15$, $25 - 4 = 21$, $36 - 9 = 27$, $49 - 16 = 33$, ...

i. What are the next several terms in this sequence?

ii. What simple expression shows how to calculate the nth term in this sequence of differences?

iii. Use algebraic reasoning to prove that the expression proposed in part ii will give the nth term in the sequence of differences.

c. **i.** Try to generalize the patterns you examined in Parts a and b. What simple expression will give the nth term in the sequence formed by calculating the differences of numbers that are k steps apart in the sequence of square numbers?

ii. Use algebraic reasoning to show that your idea is correct.

4 **Reasoning about Data** An experiment was designed to see whether a program of special stepping and foot-placing exercises for 12 minutes each day could speed up the process of babies learning to walk. As part of this study, 12 baby boys were randomly assigned to the special exercise group or to the "exercise control" group. For the control group, parents were told to make sure their infant sons exercised at least 12 minutes per day. But they were not given any special exercises to use, and they were not given any other instructions about exercise.

The researchers recorded the age, in months, when each baby first walked without help.

Age that Baby Boys Learned to Walk

Special Exercise (in months)	**Exercise Control** (in months)
9	11
9.5	10
9.75	10
10	11.75
13	10.5
9.5	15

Source: Phillip R. Zelazo, Nancy Ann Zelazo, and Sarah Kolb. "Walking in the Newborn," Science 176, 1972, pp. 314–315.

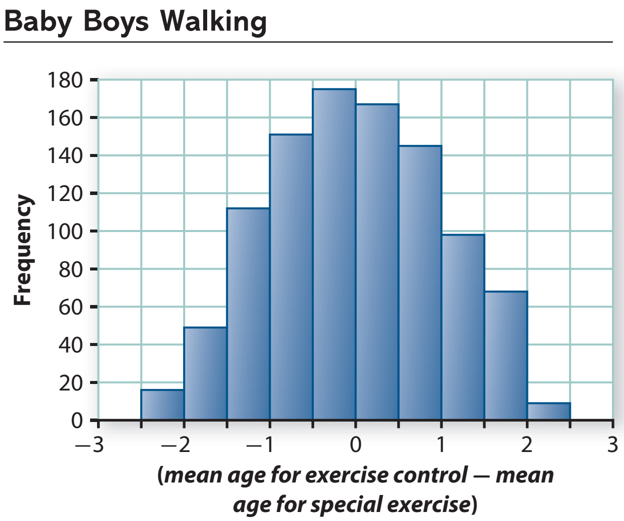
a. What are the treatments? What is the response?

b. Does this study have the three characteristics of a well-designed experiment? Could it have been double blind? Is the placebo effect a possible issue here?

c. What is the difference: *mean walking age for exercise control group − mean walking age for special exercise group*?

d. Suppose that the treatment makes no difference in the age that baby boys learn to walk. Would you expect there to be a nonzero difference in *mean walking age for exercise control − mean walking age for special exercise* for almost any two groups of six baby boys? Explain.

e. Describe how to conduct a randomization test to determine if the different treatments cause a difference in the mean age that baby boys learn to walk. Perform 10 runs and add them to the randomization distribution below, which shows the results of 990 runs.



f. What is your estimate of the probability that just by the random assignment of treatments to subjects, you get a difference that is as least as extreme as that from the real experiment? What is your conclusion about whether you have evidence from this experiment that the special exercises are better than just reminding parents to be sure their baby boys exercise at least 12 minutes per day?

Summarize the Mathematics

In this unit, you learned some basic reasoning principles and strategies that are useful in justifying claims involving concepts of geometry, algebra, and statistics.

a. How is *deductive* reasoning different from *inductive* reasoning? Why are both types of reasoning important in mathematics?

b. When trying to prove an if-then statement, with what facts do you begin? What do you try to deduce?

c. What strategies and geometric properties can you use to prove that two lines are perpendicular? That two lines are parallel? Draw and label sketches to illustrate how those properties are used.

d. How would you go about proving a statement like "The sum of the measures of the angles of a trapezoid is 360°"?

e. What overall strategy and rules of algebra can you use to prove that an equation like $2ab + c^2 = (a + b)^2$ implies that $c^2 = a^2 + b^2$?

f. If you want to see whether two algebraic expressions like $(n + 2)^2 - n^2$ and $4n + 4$ are equivalent, you could begin by comparing tables and graphs of the functions $y = (n + 2)^2 - n^2$ and $y = 4n + 4$. How would an algebraic proof give different evidence that the expressions are equivalent?

g. What are the mathematical conventions about *order of operations* when numerical patterns and relationships are represented with symbolic expressions and equations?

h. What are the differences between sample surveys, experiments, and observational studies?

i. How do you use a randomization test to determine if the result of an experiment to compare two treatments is statistically significant?

j. How is statistical reasoning similar to algebraic and geometric reasoning and how is it different?

Be prepared to share your responses and reasoning with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.