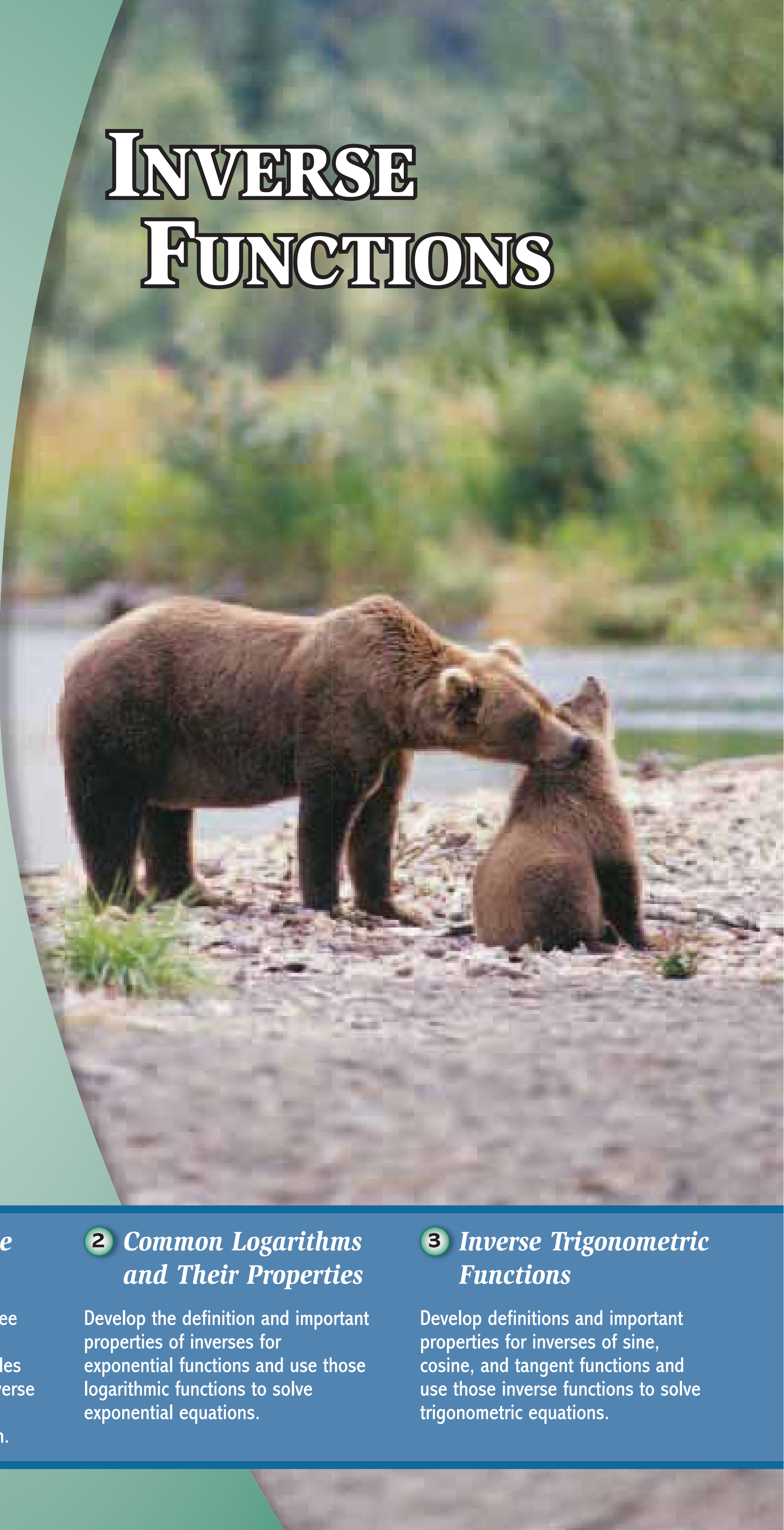
UNIT 8

INVERSE FUNCTIONS



In earlier units, you learned how polynomial, rational, exponential, and trigonometric functions can be useful tools for modeling the relationships between dependent and independent variables in a wide variety of problem situations. In many of those problems, the key challenge is finding values of the independent variable that lead to required values of the dependent variable.

For example, if you know the rate at which an endangered animal population is recovering, you certainly want to find the time when that population is likely to reach a specified safe level. Solution of problems like that is helped by calculation of the inverse for the function relating population to time.

Work on the investigations of this unit will develop your understanding and skill in use of inverse functions, especially the square root, logarithm, and inverse trigonometric functions. Key ideas are encountered in three lessons.

Lessons

1 *What Is An Inverse Function?*

Discover conditions that guarantee existence of inverse functions, develop strategies for finding rules of inverse functions, and use inverse functions to solve problems of coding and decoding information.

2 *Common Logarithms and Their Properties*

Develop the definition and important properties of inverses for exponential functions and use those logarithmic functions to solve exponential equations.

3 *Inverse Trigonometric Functions*

Develop definitions and important properties for inverses of sine, cosine, and tangent functions and use those inverse functions to solve trigonometric equations.

LESSON 1

What Is An Inverse Function?

We live in a world of instant information. Numbers, text, and graphic images stream around the globe at every hour of the day on every day of the year. For most data transmission media, information must be translated into numerical form and then into electronic signals. When the signal reaches its destination, it must be translated back into the intended text, graphic, or numerical information format.

Cell phones are one of the most popular communication devices of the electronic information era. Like standard landline telephones, they permit spoken conversations across great distances. They can also be used to send *text messages*. A study in the fall of 2007 found that about 1 billion text messages were sent each day.

Sending text via cell phones takes advantage of the fact that the standard telephone dial has letters associated with numbered buttons. When your cell phone is set in text message mode, pressing number buttons sends letters to the intended receiver.

Think About This Situation

The challenge of sending clear text messages is finding a way to translate letters into numbers so that the receiver can translate those numbers back into the intended letters. One popular procedure for text message coding, called *predictive text*, asks the sender to simply press numbers that correspond to desired letters.

a How do you think a cell phone using the predictive text procedure determines intended words from number sequences?

b Suppose that a text message is entered by pressing **4**, **6**, **6**, **3**, **2**, **2**, **5**, **5**. What do you think was the intended message?

In this lesson, you will learn how mathematical functions and their inverses are used to make accurate coding and decoding of information possible and how function inverses are used to solve other important mathematical problems.

Investigation 1 Coding and Decoding Messages

Using the predictive text procedure, the keypad of a cell phone assigns each letter a numerical code. However, this coding function cannot be reversed reliably to find the intended letters from the numbers that were transmitted. As you work on the problems of this investigation, look for an answer to the following question:

What properties of a mathematical function f make it possible to find a related function that reverses the domain, range, and individual assignments of f?

1 Suppose that f is the function that uses a standard cell phone keypad to assign number codes to letters of a text message using the predictive text procedure.

a. What number sequence would be used to send the message GOOD CALL?

b. What different messages could result from attempts to decode the number sequence produced in Part a?

c. What is the problem with predictive text coding that can lead to unclear message transmission?

d. What changes in the coding process could avoid the problems revealed in your answers to Parts b and c?

2 To avoid the problems that can occur when several letters are assigned the same numerical code on a cell phone keypad, information systems, like those used by computers, employ more than ten numbers to code the letters of the alphabet and important symbols. For example, a coding scheme might make the assignments in the next table.

Symbol	0	1	⋯	9	A	B	C	⋯	X	Y	Z	.	—
Code	0	1	⋯	9	10	11	12	⋯	33	34	35	36	37

a. Using this coding scheme, what message is implied by the sequence of code numbers 22, 14, 14, 29, 37, 22, 14, 37, 10, 29, 37, 9?

b. Why does this coding scheme involve very low risk of incorrect message transmission?

c. If you want a message to be sent so that only the intended recipient can read it, why will a simple code like that shown in the table *not* be effective?

To make messages difficult for data spies to decode, the first step of letter-to-number coding is often followed by use of further *encryption* algorithms. The aim is to disguise decoding clues like known letter frequencies and patterns of repeated letters in the English language. For example, the letter "E" has been found to occur roughly 13% of the time in English text.

3 Using the basic coding scheme from Problem 2, the message I WILL BE THERE is translated into the number sequence 18, 37, 32, 18, 21, 21, 37, 11, 14, 37, 29, 17, 14, 27, 14. Suppose that the basic symbols-to-numbers coding is followed by application of the linear function $f(x) = x + 16$ to each code number.

a. What number sequence gives the encrypted message now?

b. What decoding formula would you give to the recipient of your messages?

c. Why does this encryption method still fail to disguise decoding clues like known letter frequencies?

4 Because *shift encryption* algorithms like $f(x) = x + 16$ are easy to decode, it is tempting to try more complex encryption functions.

a. Study the patterns of code assignments made by the following functions, and explain why they also have weaknesses as encryption algorithms.

i. $g(x) = 2x + 1$

ii. $h(x) = x^2$

iii. $k(x) = 38x - x^2$

b. See if you can devise an encryption algorithm of your own that does not have the flaws of the examples in Part a.

5 Where possible, describe in words and find algebraic expressions for strategies that could be used to *decode* messages that have been encrypted by the following functions.

a. $f(x) = x + 7$

b. $g(x) = 2x + 1$

c. $h(x) = x^2$

d. $k(x) = 38x - x^2$

If an encryption algorithm assigns the same code number to every occurrence of a symbol, it can be broken easily. For example, if the letter "E" is always assigned the code number 29, code breakers will use the fact that "E" is the most common letter in English to guess that 29 stands for that letter. An encryption scheme that uses matrices avoids this problem of simpler coding algorithms.

6 One simple matrix coding scheme begins by arranging the message number codes in the form of 1×2 matrix blocks. For example, the sequence 2, 0, 11, 11, 7, 14, 12, 4 becomes:

$M_1 = \begin{bmatrix} 2 & 0 \end{bmatrix}$ $\quad M_2 = \begin{bmatrix} 11 & 11 \end{bmatrix}$ $\quad M_3 = \begin{bmatrix} 7 & 14 \end{bmatrix}$ $\quad M_4 = \begin{bmatrix} 12 & 4 \end{bmatrix}$

Next, each of those message blocks is multiplied by a coding matrix, like $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

a. Find the encrypted form of each message block: M_1C, M_2C, M_3C, M_4C. Then rewrite the message number sequence in its newly encrypted form.

b. What do you notice about the resulting pattern of numbers in the encrypted message sequence? How does this strategy appear to solve the problem of coding clues given by known letter frequencies?

c. What decoding directions would you give to the message recipient?

Summarize the Mathematics

In this investigation, you compared several encryption procedures to reveal properties of schemes that provide accurate and difficult-to-break message-coding methods. The methods you examined were:

(1) the assignment of number codes to letters by cell phone buttons.

(2) the assignment of number codes to digits, letters, periods, and spaces as in $0 \to 0$, $1 \to 1$, … , $A \to 10$, $B \to 11$, … , $Z \to 35$, $. \to 36$, $— \to 37$.

(3) the algorithm that encrypted numbers with the function $f(x) = x + 16$.

(4) the algorithm that encrypted numbers with the function $g(x) = 2x + 1$.

(5) the algorithm that encrypted numbers with the function $h(x) = x^2$.

(6) the algorithm that encrypted numbers with the function $k(x) = 38x - x^2$.

(7) the algorithm that encrypted number blocks by multiplication with $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

a Which of the coding and encryption functions assigns code numbers to message letters and numbers in ways that can always be decoded to accurately retrieve the intended message? For the methods that do *not* decode accurately, explain why not.

b What properties of a coding function f will guarantee that messages encrypted by that function can always be decoded accurately?

Be prepared to explain your ideas to the class.

Check Your Understanding

The mathematical challenge of decoding messages is similar to the problem that occurs in many other situations. For example, the student government at Rosa Parks High School was planning a fundraising project that involved selling pizza at girls' and boys' basketball games. Market research among students and parents revealed two functions relating *sales* and *profit* to *price*:

Sales: $n(x) = 250 - 80x$, where x is price in dollars per slice and $n(x)$ is number of slices sold.

Profit: $P(x) = -80x^2 + 290x - 125$, where x is price in dollars per slice and $P(x)$ is profit.

Study tables and graphs of these two functions to explain the answer to each question below.

a. Is it possible to find the number of slices sold if you know the price per slice?

b. Is it possible to find the price per slice if you know the number of slices sold?

c. Is it possible to find the profit from the project if you know the price per slice?

d. Is it possible to find the price per slice if you know the profit from the project?

Investigation 2 Finding and Using Inverse Functions

A mathematical function f sets up a correspondence between two sets so that each element of the domain D is assigned exactly one image in the range R. If another function g makes assignments in the opposite direction so that when $f(x) = y$ then $g(y) = x$, we say that g is the **inverse** of f.

The inverse relationship between two functions is usually indicated with the notation $g = f^{-1}$ or $g(x) = f^{-1}(x)$. The notation $f^{-1}(x)$ is read "f inverse of x." In this context, the exponent "-1" *does not mean* the reciprocal "one over $f(x)$." Inverses are often useful in solving problems, but there are many functions that do not have inverses. You have seen examples in your work on the coding problems of Investigation 1.

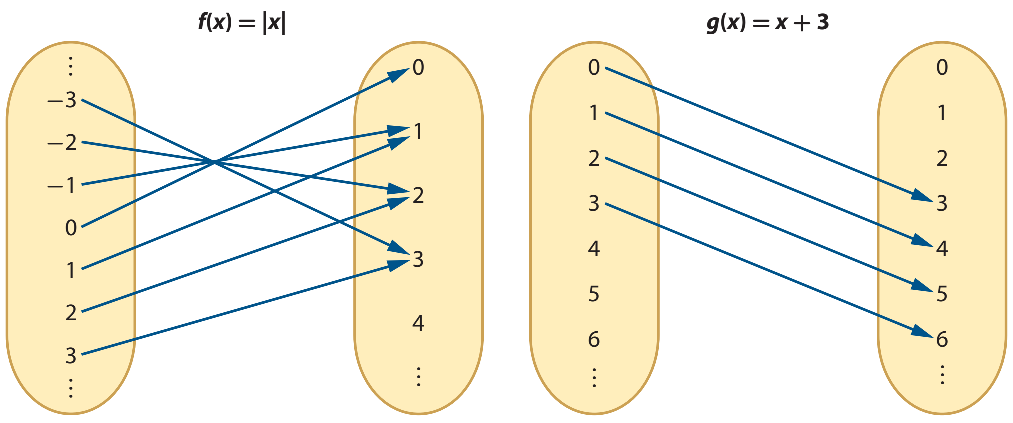
For functions with numeric domains and ranges, it is usually very helpful to describe the rule of assignment with an algebraic expression. It is also useful to find such rules for inverse functions. As you work on the problems of this investigation, look for answers to the following questions:

Which familiar types of functions have inverses?

How can rules for inverse functions be derived?

Which Functions Have Inverses? It is often helpful to think about inverse functions by using *arrow diagrams* and *coordinate graphs*.

For example, the next arrow diagrams show the patterns of assignments by two familiar functions.



1 Consider the two functions and the given (incomplete) arrow diagrams.

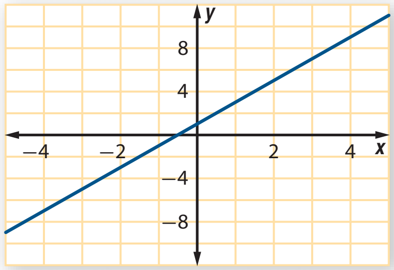
a. Does $f(x) = |x|$ have an inverse? How does the diagram show reasons for your answer?

b. Does $g(x) = x + 3$ have an inverse? How does the diagram show reasons for your answer?

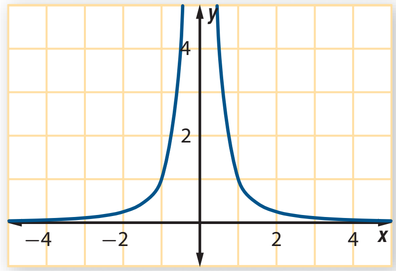
c. How can you use an arrow diagram like those shown to decide whether a function does or does not have an inverse?

2 Since functions are often represented by coordinate graphs, it helps to be able to inspect such graphs to see if inverses exist. Consider the graphs of functions given below.

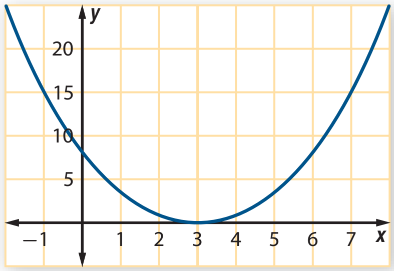
Graph I $f(x) = 2x + 1$



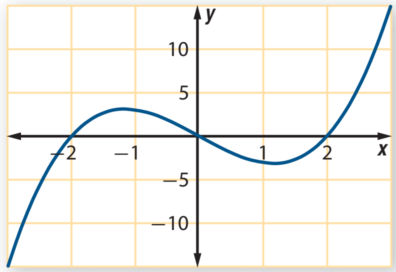
Graph II $f(x) = \frac{1}{x^2}$



Graph III $f(x) = (x - 3)^2$



Graph IV $f(x) = x^3 - 4x$



a. Which of the graphs show functions that have inverses?

b. For each function that does not have an inverse, see if you can describe a restricted (reduced) domain for the given algebraic rule so that the resulting function does have an inverse.

c. Based on your work with the four graphs, describe what you think are the key differences between graphs of functions that have inverses and graphs of those that do not.

Applications of Inverse Functions The information provided by an inverse function is often very helpful for solving practical problems in science or in business.

3 Hot air balloon flying is a popular recreation in several southwestern states. At some gatherings of balloonists, there is a competition to hit targets on the ground with bags dropped from balloons at various altitudes.

The balloons drift toward and over the target at various speeds. To have the best chance that their drop bags will hit the target, balloonists need to know how fast the bags will fall toward the ground. Principles of physics predict that the velocity in feet per second will be a function of time in seconds with rule $v(t) = 32t$. The distance fallen in feet will also be a function of time in seconds with rule $d(t) = 16t^2$.

a. How long will it take the drop bag to reach velocity of:

i. 48 feet per second?

ii. 100 feet per second?

iii. 150 feet per second?

b. What rule can be used to calculate values for the inverse of the velocity function—to find the time when any given velocity is reached?

c. How long will it take the drop bag to reach the ground from altitudes of:

i. 144 feet?

ii. 400 feet?

iii. 500 feet?

d. What rule can be used to calculate values of the inverse of the distance function—to find the time it takes for the drop bag to fall any given distance?

4 The daily profit of the Texas Theatre movie house is a function of the number of paying customers with rule $P(n) = 5n - 750$.

a. What numbers of paying customers will be required for the theater to have:

i. a profit of \$100?

ii. a profit of \$1,250?

iii. a loss of \$175?

b. What rule can be used to calculate values for the inverse of the profit function—to find the number of paying customers required to reach any particular profit target?

Finding Rules for Inverse Functions The situations in Problems 3 and 4 illustrate the main reason for being concerned about inverse functions: We often know a function that shows how to calculate values of one variable y from known values of another variable x. But problems in such situations often require discovering the value(s) of x that will produce specific target values of y.

If the function f shows how to use values of x to calculate values of y, then f^{-1} shows how to use values of y to calculate corresponding values of x. The challenge is using the rule for f to find the rule for f^{-1}.

5 For each of the following linear relationships between y and x, describe in words the calculations required to find the value of x corresponding to any given value of y.

a. $y = g(x)$ and $g(x) = 3x$

b. $y = h(x)$ and $h(x) = 3x + 5$

c. $y = j(x)$ and $j(x) = 3x - 7$

d. $y = k(x)$ and $k(x) = \frac{3}{5}x + 4$

e. $y = s(x)$ and $s(x) = mx + b$ $(m \neq 0)$

When analysis of the relationship between two variables begins with the equation $y = f(x)$, it probably seems natural to express the inverse relationship with the equation $x = f^{-1}(y)$ and to write the rule for the inverse function using y as the independent variable.

However, the convention in mathematical practice is to write the rule for f^{-1} in terms of x as well—with the understanding that the letter x stands for the independent variable in the rules for both f and f^{-1}.

6 Use this notational convention to write algebraic rules for each inverse function in Problem 5.

a. If $g(x) = 3x$, then $g^{-1}(x) = \ldots$.

b. If $h(x) = 3x + 5$, then $h^{-1}(x) = \ldots$.

c. If $j(x) = 3x - 7$, then $j^{-1}(x) = \ldots$.

d. If $k(x) = \frac{3}{5}x + 4$, then $k^{-1}(x) = \ldots$.

e. If $s(x) = mx + b$ $(m \neq 0)$, then $s^{-1}(x) = \ldots$.

7 Finding inverses (when they exist) for nonlinear functions is generally more challenging than for linear functions. Study the following strategies for finding the inverse of $f(x) = \frac{5}{x-2}$ to see if you agree that they are both correct.

Strategy I

If $\quad y = \frac{5}{x-2}$

Then $\quad y(x-2) = 5$

Then $\quad x - 2 = \frac{5}{y}$

Then $\quad x = \frac{5}{y} + 2$

So, $\quad f^{-1}(x) = \frac{5}{x} + 2.$

Strategy II

If $\quad y = \frac{5}{x-2}$

Swap the roles of y and x to get

$x = \frac{5}{y-2}$

Then $\quad y - 2 = \frac{5}{x}$

Then $\quad f^{-1}(x) = \frac{5}{x} + 2.$

Adapt one strategy or the other to find rules for inverses of the following functions.

a. If $g(x) = \frac{7}{x} + 4$, then $g^{-1}(x) = \ldots$.

b. If $h(x) = \frac{7}{x+4}$, then $h^{-1}(x) = \ldots$.

c. If $j(x) = \frac{7}{x}$, then $j^{-1}(x) = \ldots$.

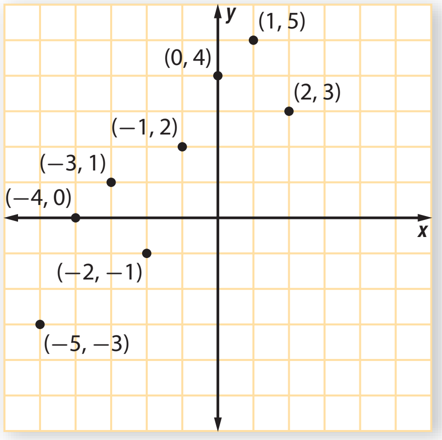
d. If $k(x) = x^2$ $(x \geq 0)$, then $k^{-1}(x) = \ldots$.

8 In some situations, you will need to find the inverse of a function that is defined only by a table of values or a graph.

a. Suppose that assignments of the function $f(x)$ are as shown in the following table. Make a similar table that shows the assignments made by $f^{-1}(x)$. Then describe the domain and range of $f(x)$ and $f^{-1}(x)$.

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	2	1	0	−1	−2	−3	−4	−5	−6

b. Suppose that the assignments of the function $g(x)$ are as shown on the graph below.



i. On a copy of that graph, plot points that represent assignments made by $g^{-1}(x)$. Then describe the domain and range of $g(x)$ and $g^{-1}(x)$.

ii. Draw line segments connecting each point (a, b) on the graph of $g(x)$ to the corresponding point (b, a) on the graph of $g^{-1}(x)$. Then describe the pattern that seems to relate corresponding points on the graphs of $g(x)$ and $g^{-1}(x)$.

Summarize the Mathematics

In this investigation, you explored properties of functions that guarantee existence of inverses and strategies for finding rules for those inverses. You also explored how the graphs of a function and its inverse are related geometrically.

a What patterns in an arrow diagram or coordinate graph for a function indicate that the function does or does not have an inverse?

b What strategies are helpful in finding the rule for f^{-1} when you know the rule for f?

c What geometric pattern relates graphs of functions and their inverses?

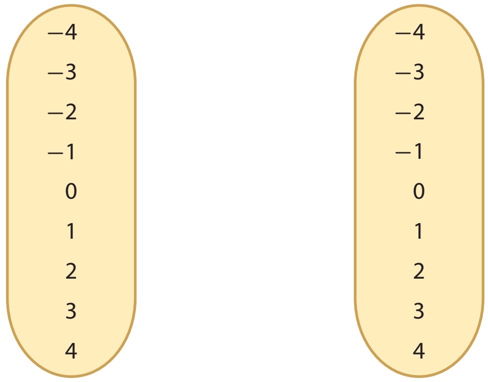
Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of functions and their inverses to complete the following tasks.

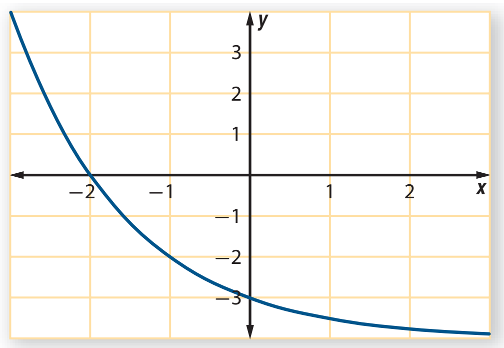
a. Draw arrows on a copy of the next diagram to illustrate image variable assignments made by the function $f(x) = -x$. Then explain how the pattern of those arrows suggests that the function does or does not have an inverse.

$f(x) = -x$

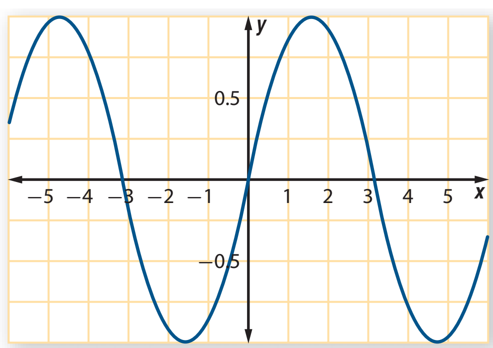


b. Which of the graphs below show functions that have inverses? Justify your response.

Graph I $f(x) = 0.5^x - 4$



Graph II $g(x) = \sin x$



c. Find algebraic rules for the inverses of the following functions.

i. $f(x) = 0.5x - 2$

ii. $g(x) = \dfrac{0.5}{x} - 2$

On Your Own

Applications

1 Suppose that you want to send the message MATH QUIZ TODAY as a cell phone text message using the predictive text procedure.

a. What sequence of numbers would you press on the standard cell phone keypad?

b. What different message might actually be received?

2 Suppose that a message was first coded by assigning 0, 1, … , 37 to numbers, letters, periods, and spaces as in Investigation 1 and then encrypted by the function $f(x) = x + 9$.

a. If the received message is given in encrypted form by 37, 23, 23, 46, 43, 33, 39, 46, 11, 46, 32, 27, 38, 23, what was the message that was sent?

b. What algebraic expression would decode any message that had been encrypted by $f(x) = x + 9$?

3 Suppose that a message was first coded by assigning 0, 1, … , 37 to numbers, letters, periods, and spaces as in Investigation 1 and then encrypted by the function $f(x) = 2x$.

a. If the received message is given in encrypted form by 38, 48, 36, 46, 74, 58, 34, 28, 74, 22, 20, 46, 26, what was the message that was sent?

b. What algebraic expression would decode any message that had been encrypted by $f(x) = 2x$?

c. Why do the encrypted message numbers make the decoding rule easy to guess?

4 Suppose that a message was first coded by assigning 0, 1, … , 37 to numbers, letters, periods, and spaces as in Investigation 1 and then encrypted by the function $f(x) = 3x + 2$.

a. If the received message is given in encrypted form by 86, 44, 71, 41, 113, 32, 113, 77, 56, 38, 89, 92, 83, 44, what was the message that was sent?

b. What algebraic expression would decode any message that had been encrypted by $f(x) = 3x + 2$?

5 Suppose that a message was first coded by assigning 0, 1, … , 37 to numbers, letters, periods, and spaces as in Investigation 1. Study patterns in code numbers assigned by the following encryption functions to see which would allow decoding of any message without confusion.

For those that are suitable, give a rule for decoding. For those that are not, explain why not.

a. $f(x) = 37 - x$

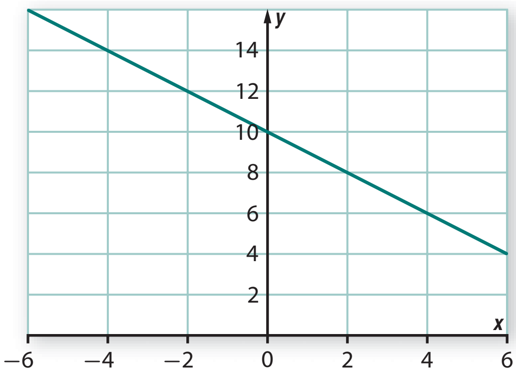
b. $g(x) = x^2 - 38x + 361$

c. $h(x) = |x - 37|$

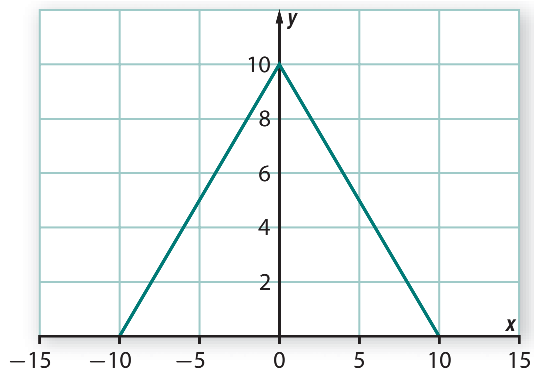
d. $k(x) = 1.5x + 0.5$

6 Which of these graphs represent functions that have inverses? Be prepared to justify each answer.

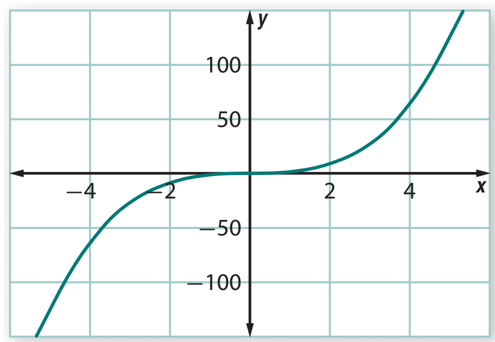
Graph I $y = 10 - x$



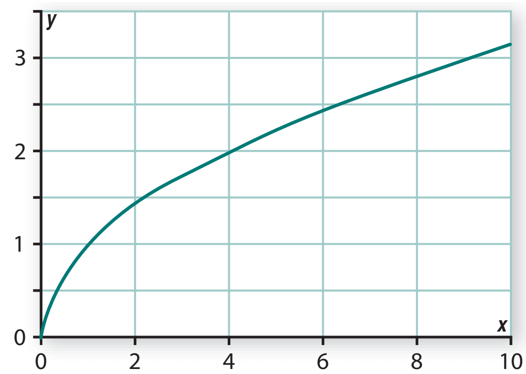
Graph II $y = 10 - |x|$



Graph III $y = x^3$



Graph IV $y = \sqrt{x}$



7 Find rules for the inverses of the following functions.

a. $f(x) = 4x - 5$

b. $g(x) = 8x^2$ (domain $x \geq 0$)

c. $h(x) = \frac{5}{x}$ (domain $x \neq 0$)

d. $k(x) = -5x + 7$

8 The following statements describe situations where functions relate two quantitative variables. For each situation:

- if possible, give a rule for the function that is described.
- determine whether the given function has an inverse.
- if an inverse exists, give a rule for that inverse function (if possible) and explain what it tells about the variables of the situation.

a. If regular gasoline is selling for $3.95 per gallon, the price of any particular purchase p is a function of the number of gallons of gasoline g in that purchase.

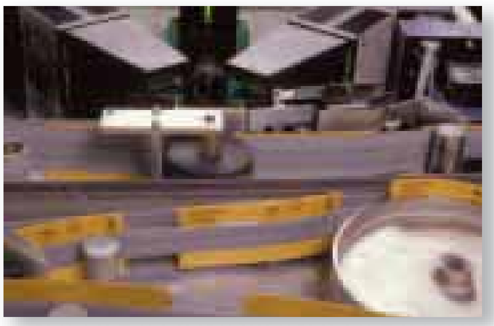
b. If a school assigns 20 students to each mathematics class, the number of mathematics classes M is a function of the number of mathematics students s in that school.

c. The area of a square A is a function of the length of each side s.

d. The number of hours of daylight d at any spot on Earth is a function of the time of year t.

Connections

9 The U.S. Postal Service uses two kinds of zip codes. One short form uses five digits, like 20906. The longer form uses nine digits, like 20742-3053.



a. How many five-digit zip codes are possible?

b. How many nine-digit zip codes are possible?

c. How do your answers to Parts a and b explain the fact that every possible U.S. mail address has a unique nine-digit zip code but not a unique five-digit code?

10 U.S. Social Security numbers all use nine-digits, generally written in three groups like 987-65-4321.

a. How many U.S. citizens can be assigned social security numbers with the current system so that each person has a unique code?

b. No social security numbers begin with the three digits 666. By how much does that restriction reduce the number of available unique social security code numbers?

c. When social security numbers were first issued, the beginning three digits of each number indicated the area office (one of eight across the U.S.) from which it was issued. How many distinct social security numbers could be issued from one such area?

11 Consider the geometric transformation with coordinate rule $(x, y) \rightarrow (x + 3, y + 2)$.

a. What kind of transformation is defined by that rule?

b. What is the rule for the inverse of that transformation?

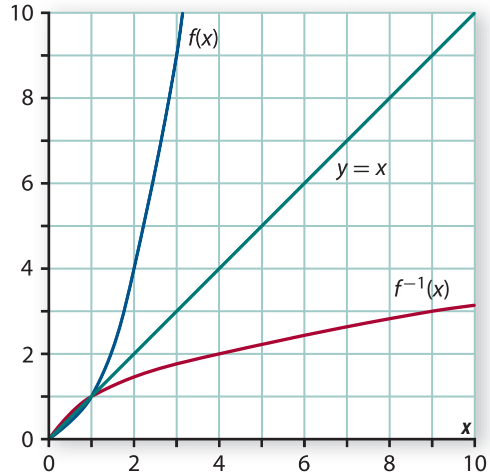
12 The arithmetic mean is a statistical function that assigns a single number to any set of numbers $x_1, x_2, \ldots, x_n$ by the general rule $\frac{\Sigma x_i}{n}$. Does this function have an inverse? Explain.

13 The formula $P = 2L + 2W$ assigns a numerical perimeter to every rectangle with length L and width W. Does this function have an inverse? Explain.

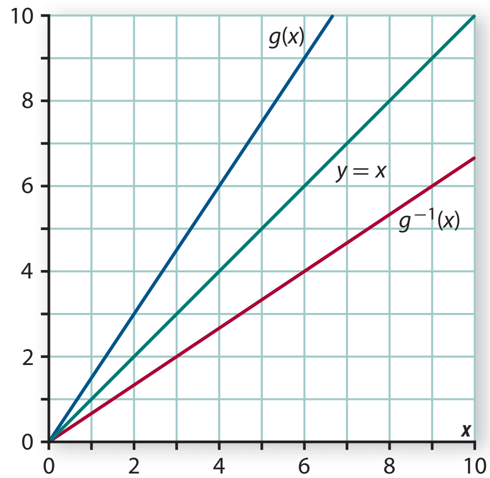
14 The following graphs show pairs of functions that are inverses of each other and the line $y = x$. For each pair of functions:

- find two points (a, b) and (c, d) on one graph and show that the points (b, a) and (d, c) are on the other graph.
- explain why the transformation $(x, y) \rightarrow (y, x)$ maps every point of one graph onto a point of the other graph.

a. $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}, x \geq 0$

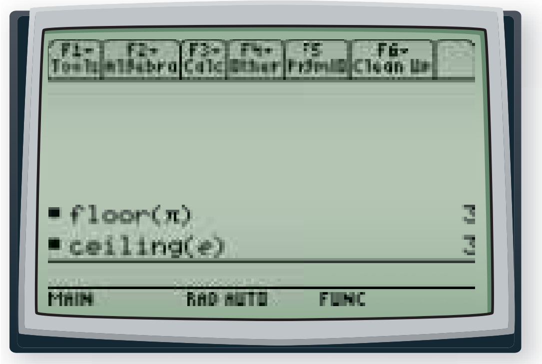


b. $g(x) = 1.5x$ and $g^{-1}(x) = \frac{2}{3}x$



15 Calculators and computers use the *floor* and *ceiling* functions in work with data. The functions are defined as follows.

- The floor function $f(x) = \lfloor x \rfloor$ assigns to any number x the greatest integer less than or equal to x. For example, $\lfloor 3.1415 \rfloor = 3$.
- The ceiling function $c(x) = \lceil x \rceil$ assigns to any number x the smallest integer greater than or equal to x. For example, $\lceil 2.71828 \rceil = 3$.



a. Explain why it is or is not possible to find the value of x when you know the value of $f(x)$.

b. Explain why it is or is not possible to find the value of x when you know the value of $c(x)$.

16 Rounding is a function used often in calculator or computer work with numeric data.

a. If $r_2(x)$ rounds every number to two decimal places, find these results from using that function.

i. $r_2(3.141)$ **ii.** $r_2(2.718)$ **iii.** $r_2(2.435)$

b. Explain why it is or is not possible to find the value of x when you know the value of $r_2(x)$.

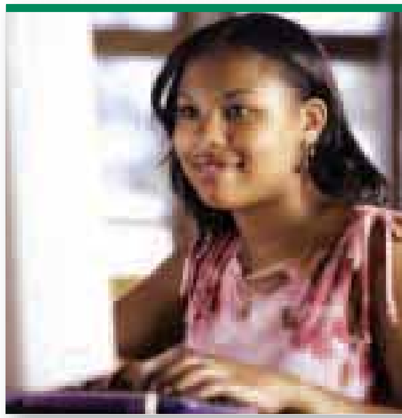
Reflections

17 In what kinds of everyday situations do you use systems that code and decode information? Explain why that coding is required and the properties that useful coding schemes must have.

18 Many computer systems require entry of passwords before allowing users to access sites like email servers and other private sources of information. What are some criteria that make sense when assigning passwords?

19 When Brianna looked up "inverse function" on the Internet, she found a sentence that stated, "A function has an inverse if and only if it is a one-to-one function." She wondered what the phrase "one-to-one" means.

> Does it mean that for each value of x there is exactly one paired value of y? Or does it mean that for each value of y there is exactly one paired value of x?



Based on your investigation of inverse functions, what do you think the one-to-one condition tells about a function?

20 The word "inverse" is used in several different ways in algebra. For example, we say that -7 is the *additive inverse* of 7 because $-7 + 7 = 0$. Similarly, we say that $\frac{7}{2}$ is the *multiplicative inverse* of $\frac{2}{7}$ because $\left(\frac{7}{2}\right)\left(\frac{2}{7}\right) = 1$.

How is the use of the word "inverse" in the phrase "inverse function" similar to its use in the phrases "additive inverse" and "multiplicative inverse"?

21 Look back at your work on Problem 8 of Investigation 1 and Connections Task 14. Given the graph of a function that has an inverse, how could you quickly sketch the graph of the inverse function? Give an example illustrating your ideas.

22 Here are two ways to think about finding the rule for an inverse function.

Brandon's Strategy: If I know the rule for $f(x)$ as an equation relating y and x, I simply swap the symbols y and x and then solve the resulting equation for y. For example, if $f(x) = 3x + 5$ or $y = 3x + 5$, then I swap y and x to get $x = 3y + 5$ and solve for y to get the rule $y = \frac{x-5}{3}$ for the inverse.

Luisa's Strategy: If I know the rule for $f(x)$ as an equation relating y and x, I solve that equation for x in terms of y and then use that equation to write the rule for the inverse of f. For example, if $f(x) = 3x + 5$ or $y = 3x + 5$, I solve for x to get $x = \frac{y-5}{3}$, that tells me the inverse function has rule $f^{-1}(x) = \frac{x-5}{3}$ or $y = \frac{x-5}{3}$.

a. Will either or both of these strategies always give the correct inverse function rule?

b. Which of the two strategies (or some other strategy of your own) makes most sense to you as a way of thinking about inverse functions and their rules?

Extensions

As you used the linear function and matrix encryption algorithms for coding, you probably noticed that the code numbers get quite large quickly. Mathematicians have devised a kind of remainder arithmetic that solves that problem. The idea is to apply the sort of counting involved in calculating with time—hours in a day or days in a week.

If you want to find the day of the week 23 days from today, you notice that 23 days is really just 2 days more than 3 weeks and count on 2 days from today. To apply this idea to simplifying code numbers, you take any number calculated with an encryption algorithm and reduce it by taking out multiples of 38. That is, you count from 0 to 37 and then start over from 0 again.

For example, suppose that you are using the encryption function $f(x) = 2x + 1$.

Step 1. The letter T is assigned the code number 29 and then $f(29) = 59$.

Step 2. If we start counting at 0 when reaching 38, the code number 59 is reduced to 21.

This method using a function and remainder arithmetic results in code numbers no larger than 37, but it must be used with some care as you will see in Extensions Tasks 23–25.

23 Simplify the result of the arithmetic of the following calculations based on the idea above.

a. $23 + 34$

b. $2(23) + 15$

c. $5(23) + 4$

d. $9(4) + 3$

e. $9(8) + 3$

f. 13^2

24 Consider the message I WILL BE THERE AT 1 that translates first into the following number sequence.

18, 37, 32, 18, 21, 21, 37, 11, 14, 37, 29, 17, 14, 27, 14, 37, 10, 29, 37, 1

a. Use the coding function $f(x) = 2x + 1$ and remainder arithmetic that starts over at 38 to find the encrypted number sequence carrying this message.

b. What problem with this encryption algorithm is revealed by coding the given message?

25 Now explore encryption by remainder arithmetic that starts over at 38 with the function $g(x) = 5x + 1$.

a. Encrypt the following message number sequence.

18, 37, 32, 18, 21, 21, 37, 11, 14, 37, 29, 17, 14, 27, 14, 37, 10, 29, 37, 1

b. Compare the result of Part a to the encryption result using $f(x) = 2x + 1$. Show that the problem encountered with that function does not occur with $g(x) = 5x + 1$, at least for this message.

c. Complete a copy of the following table to show the encrypted form of each number from 0 to 37. Explain how the pattern of those results shows that $g(x)$ does not assign the same code number to any pair of distinct letters.

Symbol	0	1	…	9	A	B	…	Z	.	—
Code Number	0	1	…	9	10	11	…	35	36	37
Encrypted	1	6	…	8	13	18	…	24	29	34

d. Apply the function $d(x) = 23x + 15$ with remainder arithmetic to each number that results from encryption by $g(x)$ in Part c. What does the pattern of results in that table suggest about the relationship of $g(x)$ and $d(x)$?

26 You have seen that without domain restrictions, quadratic functions never have inverses. Explore that same question for the case of cubic polynomials.

a. Show by analysis of the graph of $y = x^3$, that this simplest possible cubic has an inverse, the cube root function. Then sketch a graph of both the cube and cube root functions on the same coordinate diagram for $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$.

b. Give other examples (where possible) of cubic polynomials that do have inverses and some that do not.

c. The general form of a cubic polynomial is $a_3x^3 + a_2x^2 + a_1x + a_0$. Use algebraic reasoning and a graphing tool to search for patterns of coefficients in cubic functions that have inverses and patterns of coefficients in cubic functions that do not have inverses.

27 A function $f(x)$ is said to be **one-to-one** or **injective** if whenever $f(a) = f(b)$, we can conclude that $a = b$.

a. Provide justifications for each step in the following proof that the function $g(x) = 3x + 5$ is one-to-one.

Step 1. Whenever $g(a) = g(b)$, we can conclude that $3a + 5 = 3b + 5$.

Step 2. The equation in Step 1 implies that $3a = 3b$.

Step 3. The equation in Step 2 implies that $a = b$.

Step 4. Therefore, $g(x)$ is one-to-one.

b. Adapt the argument in Part a to prove that any linear function $h(x) = mx + n$ is one-to-one.

c. Prove that $q(x) = x^2$ (and in fact, any quadratic polynomial function) is not one-to-one.

d. Explain why any exponential function is one-to-one.

e. Explain why the sine and cosine functions are not one-to-one.

f. Explain why a function that is *not* one-to-one does not have an inverse.

g. Explain why a function that is one-to-one does have an inverse.

28 Any 2×2 matrix defines a transformation of points in the coordinate plane that maps points in the following general form.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

a. Show that the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ are inverses of each other.

b. Evaluate $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

c. Multiply the result from Part b by $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

29. The simplest 2×2 coding matrices are those in the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $ad - bc = 1$. In those cases, the inverse of A can be found using the formula: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

a. Use the given formula to find the inverse of $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$.

b. Prove that the given formula will always produce the inverse of such a matrix by showing that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2×2 identity matrix.

Review

30. Use properties of exponents to write each of these expressions in a different equivalent form.

a. $(x^5)(x^2)$

b. $(x^5) \div (x^2)$

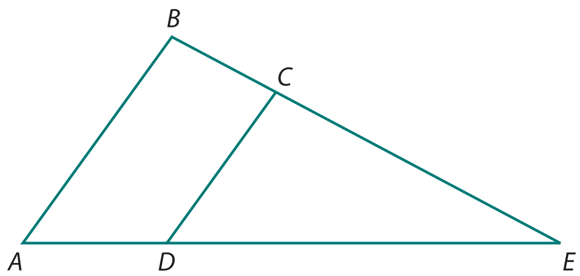
c. $(x^5)^2$

d. x^{-5}

31. In the figure at the right, $\overline{AB} \parallel \overline{CD}$.

a. Prove that $\triangle ABE \sim \triangle DCE$.

b. If $BC = 6$ cm, $CE = 9$ cm, $CD = 8$ cm, and $DE = 12$ cm, find AB, AE, and AD.



32. Consider quadrilateral $ABCD$ represented by the matrix
$\begin{bmatrix} -3 & 1 & -1 & -5 \\ 5 & 4 & 0 & -1 \end{bmatrix}$.

a. Find the image of quadrilateral $ABCD$ reflected across the line $y = x$. Call the image $A'B'C'D'$.

b. Using a coordinate grid, draw quadrilateral $ABCD$, its image $A'B'C'D'$, and the line $y = x$.

c. How is the line $y = x$ related to each segment connecting an image point of the quadrilateral with its preimage point?

33. Solve these equations for x.

a. $10{,}000 = 10^x$

b. $100{,}000 = 10^{2x + 1}$

c. $15 = 2^x + 7$

d. $2^{3x} = 8^x$

e. $2^{x^2 + 5x + 6} = 1$

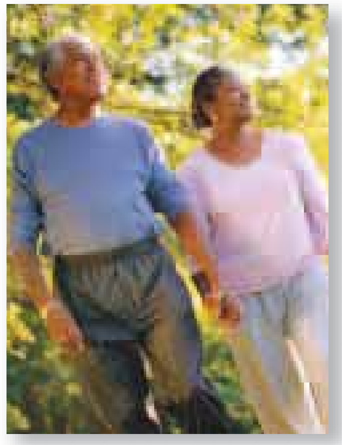
34 Suppose that Alvin and Tia are each starting a new walking program. On the first day, they will each walk 1,000 yards. Alvin will increase his distance by 100 yards each day. Tia will increase her distance by 6% each day.

a. Is the sequence of each person's daily walking distance arithmetic, geometric, or neither? Explain your reasoning.

b. Who will walk further on the 20th day? By how much?

c. Who will have walked further over the entire 20-day period? By how much?

d. Write a function rule that will give the distance each person walked on day n.



35 Write these expressions in equivalent form as products of linear factors.

a. $x^2 + 7x$

b. $x^2 + 7x + 12$

c. $x^2 + 7x - 8$

d. $x^2 - 49$

e. $x^2 - 6x + 9$

f. $3x^2 - 2x - 8$

36 Matthew is riding a bike with wheels that have a 24-centimeter radius. Suppose that he is able to pedal at a steady rate so that the wheels make 120 rotations per minute.

a. Find his angular velocity in both degrees per minute and radians per minute.

b. How many meters will he travel in 5 minutes of riding at this speed?

37 Describe the largest possible domains for functions with these rules.

a. $f(x) = \sqrt{5 - x}$

b. $g(x) = \dfrac{3x}{x - 5}$

c. $h(x) = \dfrac{3x + 5}{x^2 + 5x + 6}$

d. $k(x) = 10^x$

38 Solutions of the following equations are not integers. For each equation, find the two consecutive integers between which the solution lies.

a. $2^x = 36$

b. $5^x = 100$

c. $7^x = \dfrac{1}{50}$

d. $10^x = 0.0035$

e. $10^{2x} = 3{,}000$

f. $11^{x - 2} = \dfrac{1}{2}$

LESSON 2

Common Logarithms and Their Properties

At several points in your study of algebra and functions, you have discovered relationships between variables that are well-modeled by exponential functions. The rules of those functions have the form $f(x) = a(b^x)$, where a and b are positive numbers ($b \neq 1$). Answering questions about these functions often requires solving equations in which the unknown quantity is the exponent. For example:

- If a count shows 50 bacteria in a lab dish at the start of an experiment and that number is predicted to double every hour, then to estimate the time when there will be 10,000 bacteria in the dish, you need to solve the equation $50(2^t) = 10{,}000$.
- The Washington Nationals baseball team was purchased in 2006 for 450 million dollars. To find how long it will take for the value of this investment to reach \$1 billion, if it increases at the fairly conservative rate of 5% per year, you need to solve the equation $450(1.05^t) = 1{,}000$.

- If 500 mg of a medicine enters a hospital patient's bloodstream at noon and decays exponentially at a rate of 15% per hour, the amount remaining active in the patient's blood t hours later will be given by $d(t) = 500(0.85^t)$. To find the time when only 25 mg of the original amount remains active, you need to solve $25 = 500(0.85^t)$.

Think About This Situation

Solving equations like those in the three exponential growth and decay situations is a kind of inverse problem. In each case, you know the output of a function but need to find the corresponding input.

a. How could you estimate solutions for the given equations by exploring patterns in tables and graphs of function values?

b. How would you approach solution of each equation by algebraic reasoning?

In this lesson, you will extend your understanding and skill in using *logarithms*—a mathematical idea that helps in solving equations like those in the given examples and in many similar problems involving exponential functions.

Investigation 1 Common Logarithms Revisited

In the *Nonlinear Functions and Equations* unit of Course 2, you learned how to use common logarithms to answer questions very similar to those in the preceding Think About This Situation. The key difference is that all problems in your earlier work involved only exponential functions with base 10 like $B(t) = 50(10^{0.3t})$, $V(t) = 450(10^{0.02t})$, and $s(t) = 200(10^{-0.046t})$.

To deal with those kinds of problems, you learned that mathematicians have developed procedures for finding exponents.

If $10^x = y$, then x is called the **base 10 logarithm** of y.

This definition of base 10 or *common* logarithms is usually expressed in function notation as:

$\log_{10} y$ or simply $\log y$, read "log base 10 of y" or "log of y"

Graphing calculators have a LOG key that automatically finds the required exponent values.

Work on the problems of this investigation will review and extend your understanding of logarithms to consider this question:

How can logarithms be used to solve equations involving exponential functions with base 10?

1. Without use of a calculator or computer **log** function, find each of the following logarithms. Be prepared to explain how you know that your answers are correct.

   a. $\log 10^{2.5}$
   b. $\log 100$
   c. $\log 10{,}000$
   d. $\log 0.01$
   e. $\log(-0.001)$
   f. $\log(10^2 \cdot 10^5)$

2. Without use of a calculator or computer **log** function, find the consecutive integers just below and just above the exact values of these logarithms. Be prepared to explain how you know that your answers are correct.

   a. $\log 25$
   b. $\log 314$
   c. $\log 3.14$
   d. $\log 0.005$

3. Use what you know about logarithms as necessary to find exact solutions for these equations.

   a. $10^x = 100$
   b. $10^{x+2} = 100$
   c. $10^{3x+1} = 100$
   d. $3(10)^{x+3} = 300$
   e. $2(10)^x = 600$
   f. $10^{2x} = 500$
   g. $10^{3x+1} = 43$
   h. $42(10)^{3x+2} = 840$
   i. $5(10)^{x+3} + 9 = 44$

4. Answer these questions related to the Think About This Situation on the previous page by using the helpful translation of the functions into exponential form with base 10.

   a. Suppose that the number of bacteria in a lab dish at any time t hours after the start of an experiment is given by the function $B(t) = 50(10^{0.3t})$. Show how to use logarithms to find the time when there will be 10,000 bacteria in the lab dish.

   b. Suppose that the value (in millions of dollars) of the Washington Nationals baseball team at any time t years after 2006 is given by the function $V(t) = 450(10^{0.02t})$. Show how to use logarithms to find the time when that investment will be worth \$1 billion.

   c. Suppose that the amount of medicine (in mg) active in a patient's blood at any time t hours after an injection is given by $m(t) = 500(10^{-0.071t})$. Show how to use logarithms to find the time when only 25 mg of the medicine remain active.

Summarize the Mathematics

In this investigation, you reviewed the definition of the base 10 or common logarithm function. You then considered the ways that this function can be used to solve inverse problems involving exponential functions with base 10.

a How would you explain to someone who did not know about logarithms what the expression $\log b = a$ tells about the numbers a and b?

b What can be said about the value of $\log y$ in each case below? Give brief justifications of your answers.

i. $0 < y < 1$
ii. $10 < y < 100$
iii. $0.1 < y < 1$
iv. $1 < y < 10$
v. $100 < y < 1{,}000$
vi. $0.01 < y < 0.1$

c Describe the main steps in solving equations in these forms for x.

i. $10^{ax+b} = c$
ii. $k(10^{ax+b}) = c$

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of common logarithms to help complete the following tasks.

a. Find these common (base 10) logarithms without using technology.

i. $\log 1{,}000{,}000$ **ii.** $\log 0.001$ **iii.** $\log 10^{3.2}$

b. Use the function $y = 10^x$, but not the **log** function of technology, to estimate each of these logarithms to the nearest integer. Explain how you arrived at your answers.

i. $\log 85$ **ii.** $\log 850$ **iii.** $\log 8.5$

c. In 2007, the U.S. Department of Interior removed grizzly bears from the endangered species list in Yellowstone National Park. The population of grizzly bears in the park at any time t years after 2007 can be estimated by the function $P(t) = 500(10^{0.02t})$. Use logarithms to find the time when this model predicts a grizzly population of 750.

Investigation 2 Covering All the Bases

Logarithms are useful for solving equations in the form $k(10^{ax + b}) = c$. However, many of the functions that you have used to model exponential growth and decay have not used 10 as the base. On the other hand, it is not too hard to transform any exponential expression in the form b^x into an equivalent expression with base 10. As you work on the problems of this investigation, look for an answer to the following question:

How can common logarithms help in finding solutions of all exponential equations?

1 Use common logarithms to express each of the following numbers as a power of 10. For example, $15 = 10^k$ when $k \approx 1.176$.

a. $2 = 10^k$ when $k = \ldots$
b. $5 = 10^k$ when $k = \ldots$
c. $1.0114 = 10^k$ when $k = \ldots$
d. $25 = 10^k$ when $k = \ldots$
e. $250 = 10^k$ when $k = \ldots$
f. $0.003 = 10^k$ when $k = \ldots$

2 Use your results from Problem 1 and what you know about properties of exponents to show how each of these exponential expressions can be written in equivalent form like $(10^k)^x$ and then 10^{kx}.

a. 2^x
b. 5^x
c. 1.0114^x

3 Use your results from Problems 1 and 2 to solve these exponential equations. Check each solution. Be prepared to explain your solution strategy.

a. $2^x = 3.5$
b. $5(2^n) = 35$
c. $5(2^t) + 20 = 125$
d. $5^p = 48$
e. $3(5^r) + 12 = 60$
f. $300(0.9^v) = 60$

4 Use the strategies you have developed from work on Problems 1–3 to solve these problems about exponential growth and decay. In each case, it will probably be helpful to write an exponential growth or decay function with the base suggested by problem conditions. Next, transform the rule for that function into an equivalent form with exponential base 10. Then solve the related equation.

a. If the world population at the beginning of 2008 was 6.7 billion and growing exponentially at a rate of 1.16% per year, in what year will the population be double what it was in 2008?

b. If 500 mg of a medicine enters a hospital patient's bloodstream at noon and decays exponentially at a rate of 15% per hour, when will only 10% of the original amount be active in the patient's body?

c. If the average rent for a two-bedroom apartment in Kalamazoo, Michigan is currently \$750 per month and increasing at a rate of 8% per year, in how many years will average rent for such apartments reach \$1,000 per month?

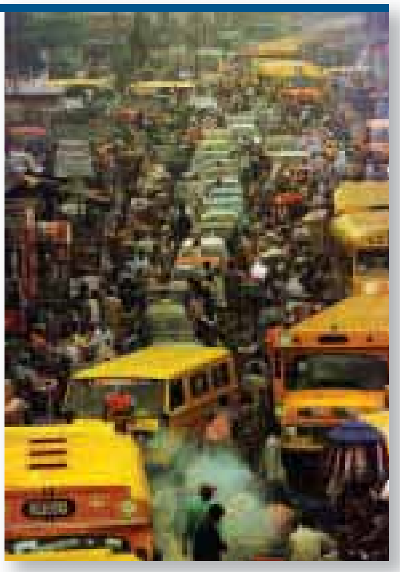
Summarize the Mathematics

In this investigation, you learned how to use logarithms to solve equations related to exponential functions with any base, $b > 0$.

a How can any exponential function with rule in the form $f(x) = b^x$ ($b > 0$) be written in an equivalent form using 10 as the base for the exponential expression?

b How can logarithms be used to solve any equation like $a(b^x) = c$ ($b > 0$)?

Be prepared to explain your ideas to the class.



Check Your Understanding

Use logarithms and other algebraic reasoning as needed to complete these tasks.

a. Solve each equation.

i. $3^x = 243$ **ii.** $8(1.5^x) = 200$ **iii.** $8x^2 + 3 = 35$

b. The population of Nigeria at the beginning of 2008 was about 140 million and growing exponentially at a rate of about 2.4% per year. (Source: www.cia.gov/library/publications/the-world-factbook/geos/ni.html)

i. What function $P(t)$ will give the population of Nigeria in millions in year 2008 + t, assuming that the growth rate stays at 2.4% per year?

ii. According to current trends, when is the Nigerian population predicted to reach 200 million? Explain how to estimate the answer to this question with a table or a graph of $P(t)$. Show how to find an "exact" answer using logarithms and other algebraic reasoning.

Investigation 3 Properties of Logarithms

To use logarithms in reasoning about problems involving exponential functions, it helps to understand key features and properties of the function $f(x) = \log_{10} x$. As you work on the problems of this investigation, look for answers to these questions:

What are the important patterns in tables and graphs for the logarithm function?

How can properties of logarithms be used to write algebraic expressions in useful equivalent forms?

Remember the relationship between logarithms and exponents:

$$10^r = s \text{ if and only if } r = \log s.$$

1 Consider the domain and range of the function $g(x) = 10^x$.

a. What is the domain of $g(x) = 10^x$; that is, for what values of x can you calculate 10^x?

b. What is the range of $g(x) = 10^x$; that is, what are the possible values of 10^x?

2 What are the domain and range of $f(x) = \log x$?

For both the domain and range, explain:

- how your answer is shown by patterns in tables of values for $f(x) = \log x$.
- how your answer is shown by patterns in the graph of $f(x) = \log x$.
- how your answer can be explained logically, using the relationship of the logarithmic function $f(x) = \log x$ and the exponential function $g(x) = 10^x$.

3 The logarithmic function $f(x) = \log x$ is the *inverse* of the exponential function $g(x) = 10^x$. Complete the following sentences to illustrate that relationship between the two functions. Be prepared to justify your claims.

a. $10^3 = 1{,}000$, so $\log 1{,}000 =$ ___.

b. $10^{1.5} \approx 31.6$, so $\log 31.6 \approx$ ___.

c. $\log 0.01 = -2$, so $10^{-2} =$ ___.

d. $\log 125 \approx 2.097$, so $10^{2.097} \approx$ ___.

4 Sketch graphs of the logarithmic function $f(x) = \log x$ and the exponential function $g(x) = 10^x$. Explain how the shape and relationship of those graphs illustrates the fact that the two functions are inverses of each other.

5 Use what you know about logarithms to evaluate these expressions.

a. $\log 10^6$ **b.** $10^{\log 100}$ **c.** $\log 10^{0.0124}$

d. $10^{\log 8.5}$ **e.** $\log 10^w$ **f.** $10^{\log z}$

6 The close connection between logarithmic and exponential functions can be used to derive some useful rules for working with algebraic expressions involving logarithms. Use what you know about exponents to write each of these expressions in a different but equivalent form.

a. $10^m 10^n$ **b.** $10^m \div 10^n$ **c.** 10^0

d. $(10^m)^n$ **e.** $1 \div 10^m$ **f.** $10^{(m-n)}$

7 Which of the following claims are true? Explain each response.

a. $\log (10^m 10^n) = m + n$ **b.** $\log (10^m \div 10^n) = m - n$

c. $\log (10^m + 10^n) = m + n$ **d.** $\log ((10^m)^n) = mn$

e. $\log (1 \div 10^m) = -m$ **f.** $\log 1 = 0$

8 You have seen that every positive number can be written as a power of 10. For example,

$$5 = 10^{\log 5}, \text{ or } 5 \approx 10^{0.7}$$
$$\text{and } 0.2 = 10^{\log 0.2}, \text{ or } 0.2 \approx 10^{-0.7}.$$

There are several other properties of the logarithmic function that are useful in transforming expressions to equivalent forms. Complete the sentences that describe each property below. For Part a, provide a reason for each step in the justification. Then adapt the reasoning in Part a to justify the reasoning in Parts b–e.

a. For any positive numbers s and t, $\log st = \log s + \log t$.

This property states that the ***logarithm of a product*** *of two numbers is equal to …*

This is true because:

$$st = 10^{\log s} \cdot 10^{\log t} \quad (1)$$
$$st = 10^{\log s + \log t} \quad (2)$$
$$\log st = \log s + \log t \quad (3)$$

b. For any positive numbers s and t, $\log \frac{s}{t} = \log s - \log t$.

This property states that the ***logarithm of a quotient*** *of two numbers is equal to …*

This is true because …

c. For any positive number s and any number t, $\log s^t = t \log s$.

This property states that the ***logarithm of a power*** *of a number is equal to …*

This is true because …

d. Another way to see why the property in Part c holds is to look at an example like $\log 5^3 = \log (5 \cdot 5 \cdot 5)$, which by the ________________________ property is equal to …

e. For any positive number t, $\log \frac{1}{t} = -\log t$.

This property states that the ***logarithm of the reciprocal*** *of a number is equal to …*

This is true because … (*Hint:* Recall that $\frac{1}{t} = t^{-1}$.)

9 Use the facts that $\log 2 \approx 0.3$ and $\log 5 \approx 0.7$ and the properties of logarithms that you proved in Problem 8 to estimate the values of these expressions without the use of technology.

a. $\log 4$ **b.** $\log 20$ **c.** $\log 8$

d. $\log 25$ **e.** $\log \frac{5}{4}$ **f.** $\log 625$

10 As with properties of exponents, there are some common errors when people use the properties of logarithms. Find the errors in the following calculations. Explain why you think the error occurs and how you would help someone see the error of his or her thinking.

a. $\log (12 + 17) = \log 12 + \log 17$ **b.** $\log 0 = 0$

c. $\log 5^3 = \log 15$ **d.** $\log (7 \times 5) = (\log 7)(\log 5)$

e. $\frac{\log 20}{\log 2} = \log 10$

11 Properties of logarithms can be used to solve exponential equations in a different way than what you developed in Investigation 2.

a. Explain how properties of logarithms are used in this sample equation solution. Then adapt the reasoning to solve the equations in Parts b–e.

$$\text{If } 5^{2x+3} = 48, \text{ then } \log 5^{2x+3} = \log 48. \quad (1)$$

$$(2x+3)\log 5 = \log 48 \quad (2)$$

$$2x + 3 = \frac{\log 48}{\log 5} \quad (3)$$

$$2x = \frac{\log 48}{\log 5} - 3 \quad (4)$$

$$x = \left(\frac{\log 48}{\log 5} - 3\right) \div 2 \quad (5)$$

b. $3^x = 25$

c. $5^{x+4} = 35$

d. $1.5^{2x+1} = 12$

e. $7(3^{0.5x}) = 42$

Summarize the Mathematics

In this investigation, you learned properties of logarithms that can be used to write logarithmic and exponential expressions in useful equivalent forms.

a Why are the functions $f(x) = 10^x$ and $g(x) = \log x$ inverses of each other?

b Rewrite each of these expressions in an equivalent form.

i. $\log (pn)$ **ii.** $\log (p \div n)$ **iii.** $\log (n^p)$

c Explain how properties of logarithms can be used to solve equations like $a(b^{kx}) = c$.

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of properties of logarithms to help complete these tasks.

a. Use properties of logarithms to write simpler expressions that are equivalent to these forms.

i. $\log (10^5 10^3)$ **ii.** $\log (10^5 \div 10^3)$ **iii.** $\log (10^5)^3$

b. Use properties of logarithms to write expressions equivalent to these in what seem to you to be simplest possible forms.

i. $\log x^5x^3$ **ii.** $\log (x^5 \div x^3)$ **iii.** $\log m - \log n$

iv. $\log m + \log n$ **v.** $m \log p + n \log p$ **vi.** $\log 10^3 + \log m^3$

c. Use properties of logarithms to solve $25(1.5^{3x}) = 1{,}000$.

On Your Own

Applications

1 Find these common (base 10) logarithms without using technology.

a. $\log 100{,}000$ **b.** $\log 0.001$ **c.** $\log 10^{4.75}$

2 Use symbolic reasoning and the definition of logarithms to solve the following equations.

a. $\log x = 2$ **b.** $10^{x-5} = 60$ **c.** $5(10)^{2x} = 60$

3 Use what you know about logarithms to solve these equations.

a. $10^x = 49$ **b.** $10^x = 3{,}000$ **c.** $10^{3x} = 75$

d. $10^{2x+1} = 123$ **e.** $15(10)^{5x+3} = 1{,}200$ **f.** $5(10)^{x+3} + 12 = 47$

4 The income required for a family of four to stay out of poverty in the United States at some time in the future is predicted by the function $I(t) = 21{,}200(10^{0.01t})$, where t is the time in years after 2008. Use logarithms to find the time when the poverty level income will be \$25,000. (Source: aspe.hhs.gov/poverty/08poverty.shtml)

5 Radioactive iodine is a dangerous by-product of nuclear explosions, but it decays rather rapidly. Suppose that the function $R(t) = 6(10^{-0.038t})$ gives the amount in a test sample remaining t days after an experiment begins. Use logarithms to find the half-life of the substance.

6 Use the definition of a common logarithm to express each of the following numbers as a power of 10.

a. $0.7 = 10^k$ when $k = \ldots$ **b.** $7 = 10^k$ when $k = \ldots$

c. $70 = 10^k$ when $k = \ldots$ **d.** $700 = 10^k$ when $k = \ldots$

7 Use your results from Applications Task 6 and what you know about properties of exponents to show how each of these exponential expressions can be written in equivalent form like $(10^k)^x$ and then 10^{kx}.

a. 0.7^x **b.** 7^x

c. 70^x **d.** 700^x

8 Use your results from Applications Tasks 6 and 7 to solve these exponential equations. Check each solution. Be prepared to explain your solution strategy.

a. $7^x = 3.5$ **b.** $5(7^n) = 35$ **c.** $5(7^t) + 20 = 125$

d. $70^p = 48$ **e.** $3(7^r) + 120 = 600$ **f.** $300(0.7^v) = 60$

9 A vacuum pump attached to a chamber removes 5% of the gas in the chamber with each pump cycle.

a. What function shows the percent of gas remaining in the chamber after n pump cycles?

b. How many full cycles are needed before at least 99% of the gas is removed?

10 A light filter lets 40% of the light that hits it pass through to the other side.

a. What function shows the fraction of light intensity allowed through by n filters?

b. How many filters are needed to reduce the light intensity to:

i. 16% of the original intensity?

ii. less than 5% of the original intensity?

iii. less than 1% of the original intensity?

11 Use the facts that $\log 20 \approx 1.3$ and $\log 16 \approx 1.2$ and the properties of logarithms to find approximate decimal values for each of these calculations—without the use of technology.

a. $\log 320$ **b.** $\log 1.25$ **c.** $\log 400$

12 Use properties of logarithms to write the following expressions in different equivalent forms.

a. $\log 3x$ **b.** $\log 5x^3$ **c.** $\log\left(\frac{7x}{5y}\right)$

d. $\log\left(\frac{1}{3x}\right)$ **e.** $\log 7 + \log x$ **f.** $3 \log y$

g. $\log x - \log 3y$ **h.** $\log 7x^3y^2$ **i.** $\log\left(\frac{7 + x}{49 - x^2}\right)$

13 Use what you know about logarithms to solve these equations without rewriting the exponential expressions in equivalent base 10 form.

a. $2^x = 10{,}000$ **b.** $3^{x + 3} = 10{,}000$ **c.** $4^{2x + 3} = 100{,}000$

d. $5(2)^x + 7 = 42$ **e.** $5(3)^{x + 1} = 2{,}500$ **f.** $1.8^{3x} = 75$

Connections

14 The time it takes a computer program to run increases as the number of inputs increases. Three different companies wrote three different programs, A, B, and C, to perform the same calculation. The time, in milliseconds, it takes to run each of the programs when given n inputs is given by the three functions below.

$$A(n) = 10{,}000 + 2 \log n$$
$$B(n) = 100 + 4n^2$$
$$C(n) = (0.00003)(10^n)$$

a. Which program, A, B, or C, takes the least amount of time for 1 input?

b. Which program is most efficient for 300 inputs?

c. Which program is most efficient for 100,000 inputs?

15 Suppose a house was purchased five years ago for \$200,000. It just sold for \$265,000. Assume that this pattern of growth has been exponential at a constant annual percent rate of change.

a. What function will give the price at any time t years from now? (*Hints:* What equation relates the two house values, the five-year time interval of appreciation in value, and the percent rate of increase in value? You can estimate solutions to equations like $R^5 = k$ by studying tables or graphs of $y = x^5$.)

b. What will be the price of the house next year?

c. When will the price of the house be \$1 million?

16 By measuring the decay of radioactive carbon-14, scientists can estimate the age of the remains of living things. Carbon-14 decays at a rate of 0.0121% per year (or retains 99.9879% of its radioactivity).

a. Write a rule for the function that gives the amount of a 5-mg lab sample remaining t years after its mass is first measured.

b. Find the half-life of carbon-14.

17 If $y = 5(3^x)$, then a table of sample (x, y) values for this function will look like the one below.

x	−2	−1	0	1	2	3	4	5	6	7
y	$\frac{5}{9}$	$\frac{5}{3}$	5	15	45	135	405	1,215	3,645	10,935

a. Add a row to a copy of this table with values $z = \log y$. For example, if $x = 2$, then $y = 45$ and $z \approx 1.65$.

b. Explain how the table pattern shows that z is a linear function of x.

c. Use properties of logarithms to write $\log 5(3^x)$ in an equivalent form that allows you to write a rule relating z to x in the form $z = mx + b$. Then compare the values produced by this linear function to those in row three of your table. Explain why the similarity occurs.

18 A large number like 2,364,700 is written in scientific notation as 2.3647×10^6 and a small number like 0.000045382 as 4.5382×10^{-5}.

a. Write each of the following numbers in scientific notation with five significant digits (rounding appropriately where necessary to meet this condition).

i. 47,265

ii. 584.73

iii. 97,485,302

iv. 0.00235148

b. Suppose the only calculator that you had was one that could multiply and divide numbers between 1 and 10 but no others. Explain how you could still use this calculator to find these products.

i. $584.73 \times 97{,}485{,}302$

ii. $47{,}265 \times 0.002351$

iii. $47{,}265 \div 584.73$

19 Logarithms were invented, in part, to help with multiplication and division of very large and very small numbers in the time long before electronic calculators. When combined with use of scientific notation, they provided a powerful tool for the sort of arithmetic required by calculations in astronomy, chemistry, and physics.

Mathematicians in the early seventeenth century produced tables of logarithms with great precision. Use what you know about scientific notation, logarithms, and the given logarithm values to do the following indicated calculations.

a. Show how the facts that $\log 1.2345 \approx 0.09149$, $\log 6.7890 \approx 0.83181$, and $\log 8.3810 \approx 0.92330$ imply that $12{,}345 \times 67{,}890 \approx 838{,}100{,}000$.

b. Show how the facts given in Part a imply that $1{,}234{,}500 \times 67{,}890{,}000 \approx 83{,}810{,}000{,}000{,}000$.

c. Use a graphing calculator to check the calculations in Parts a and b, and record the calculator outputs. Explain how the forms of those outputs tell the same story as your work with logarithms and scientific notation.

20 The common logarithm function has the very useful special property that for any positive numbers a and b, $\log ab = \log a + \log b$. Mathematicians say that the logarithm function satisfies the general functional equation $f(ab) = f(a) + f(b)$. Which, if any, of the following familiar functions satisfy that functional equation? Give proofs of those that do and counterexamples for those that do not.

a. $f(x) = 3x$ (To begin, consider: Does $3(ab) = 3a + 3b$ for all pairs (a, b)?)

b. $f(x) = x + 5$

c. $f(x) = x^2$

21 Another interesting type of functional equation asks whether $f(ab) = f(a)f(b)$ for all pairs (a, b). Which, if any, of the following functions satisfy that multiplicative functional equation? Give proofs of those that do and counterexamples for those that do not.

a. $f(x) = 3x$

b. $f(x) = x + 5$

c. $f(x) = x^2$

d. $f(x) = \frac{1}{x}$, $x \neq 0$

22 Another interesting type of functional equation asks if $f(a + b) = f(a) + f(b)$ for all pairs (a, b). Which, if any, of the functions in Task 21 satisfy that additive functional equation?

Reflections

23 Suppose that n is a positive number.

a. If $0 < \log n < 1$, what can you say about n?

b. If $5 < \log n < 6$, what can you say about n?

c. If $p < \log n < p + 1$, where p is a positive integer, what can you say about n?

24 Students learning about logarithms often find it helpful to deal with the new idea by frequent reminders that *logarithms are really just exponents*.

a. Why is it correct to say that $\log 20 \approx 1.3010$ is an exponent?

b. How does the phrase "logarithms are really just exponents" help in finding:

i. $\log 1{,}000$

ii. $\log 0.001$

25 The logarithmic function $f(x) = \log x$ is the inverse of the exponential function $g(x) = 10^x$. How can you use this fact and a graph of $g(x) = 10^x$ to quickly sketch the graph of $f(x) = \log x$ for $0 < x < 10{,}000$?

26 For $x > 1$, values of the logarithmic function $f(x) = \log x$ increase very slowly while values of the exponential function $g(x) = 10^x$ increase very rapidly. What is it about the definitions of those two functions that causes this contrasting behavior?

27 How do the properties of exponents and the phrase "logarithms are really just exponents" help in explaining these properties of the logarithm function?

a. $\log ab = \log a + \log b$

b. $\log (a \div b) = \log a - \log b$

c. $\log 1 = 0$

d. $\log a^n = n \log a$

28 With the introduction of logarithmic functions, you are now able to solve exponential equations using algebraic reasoning. Solve the equations below using each of the listed strategies. Then compare your solutions and the ease with which each method produces solutions or accurate estimates of solutions.

- Algebraic reasoning
- Approximation using function graphs
- Approximation using tables of function values

a. $100 = 4.5x - 885$

b. $3x^2 + x + 12 = 14$

c. $3(1.2^t) = 14$

29 What are the advantages of finding solutions to exponential equations by estimation using tables and graphs of exponential functions versus using algebraic reasoning with logarithms?

Extensions

30 Use properties of exponents and logarithms to solve the following equations.

a. $\log 10^x = 4$

b. $2^{2x+2} = 8^{x+2}$

31 Recall that a prime number n is an integer greater than 1 that has only 1 and n as divisors. The first eight primes are 2, 3, 5, 7, 11, 13, 17, and 19. Mathematicians have proven that the number of primes less than or equal to n is approximated by $\frac{0.4343n}{\log n}$. This formula is an amazing discovery since the primes appear irregularly among the natural numbers.

a. Count the actual number of primes less than or equal to n to complete a copy of the table below. Plot the resulting (n, *number of primes* $\leq n$) data.

n	10	25	40	55	70	85	100	115	130	145
Number of Primes $\leq n$	4				19	23	25	30	31	34

b. Graph $P(n) = \frac{0.4343n}{\log n}$, $0 < n \leq 150$. How well does this function model the counts in Part a?

c. Use the function $P(n)$ to estimate the number of primes less than or equal to 1,000. Less than or equal to 1,000,000. Less than or equal to 10^{18}.

d. According to this $P(n)$, about what percent of the numbers up to 10^6 are prime? Up to 10^{18}?

32 When you go to the movies, the number of frames that are displayed per second affects the "smoothness" of the perceived motion on the screen. If the frames are displayed slowly, our minds perceive the images as separate pictures rather than fluid motion.

However, as the frequency of the images increases, the perceived gap between the images decreases and the motion appears fluid. The frequency f at which we stop seeing a flickering image and start perceiving motion is given by the equation $f = K \log S$, where K is a constant and S is the brightness of the image being projected.

a. S is inversely proportional to the square of the observer's distance from the screen. What would be the effect on f if the distance to the screen is cut in half? What if the distance to the screen is doubled?

b. If the image is being projected at a slow frequency and you perceive a flicker, where should you move in the theater: closer to the screen or farther from the screen?

c. Suppose the show is sold out and you cannot move your seat. What could you do to reduce the flickering of the image on-screen?

33 It is possible to define logarithms for any positive base b as follows.

$$\log_b m = n \text{ if and only if } m = b^n$$

Justify each step in the following proof that $\log_b m = \frac{\log_{10} m}{\log_{10} b}$.

Suppose that $m = b^n$.

Then $\log_{10} m = n \log_{10} b$. (1)

$\log_{10} m = (\log_b m)(\log_{10} b)$ (2)

$\log_b m = \frac{\log_{10} m}{\log_{10} b}$ (3)

Review

34 Consider the function $f(x) = (x^2 - 4x)(x + 3)$.

a. Determine the value of $f(-2)$.

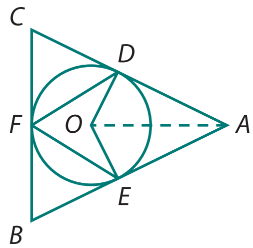
b. Solve $f(x) = 0$.

c. Rewrite $f(x)$ in standard polynomial form. Identify the degree of $f(x)$.

d. For what values of x is $f(x) < 0$?

e. Estimate the coordinates of all local maximum and minimum points of $f(x)$.

35 In the diagram at the right, the circle with center O is inscribed in $\triangle ABC$, the radius of the circle is 6, and $m\angle DAE = 52°$. Determine each of the following measures.



a. $m\angle DOE$

b. $m\overset{\frown}{DE}$

c. $m\angle DFE$

d. AD

36 Write each product or sum of rational expressions in equivalent form as a single algebraic fraction. Then simplify the result as much as possible.

a. $\dfrac{x^3}{x+2} \cdot \dfrac{x+4}{x}$

b. $\dfrac{3x}{x+4} \cdot \dfrac{x^2-16}{x^2}$

c. $\dfrac{6}{x+3} + \dfrac{4x+6}{x+3}$

d. $\dfrac{x}{5} + \dfrac{x+1}{10}$

37 Complete the following for each recursive definition of a sequence.

- Write the first five terms of the sequence.
- Determine if the sequence is arithmetic, geometric, or neither.
- If it is arithmetic or geometric, write a function formula for a_n.
- If it is arithmetic or geometric, find S_{12}.

a. $a_n = a_{n-1} - 3$, $a_0 = 50$

b. $a_n = 2a_{n-1} + 6$, $a_0 = 12$

c. $a_n = \dfrac{a_{n-1}}{2}$, $a_0 = 256$

38 How is the shape of an exponential function graph related to the base of the exponent?

39 Plans for the installation of a communication tower call for attaching a 200 feet long support wire to a point that is 125 feet above the ground. What is the degree measure of the angle formed by the support wire and the ground?

40 Write each of the following radicals in equivalent form with smallest possible numbers and simplest possible expressions under the radical sign.

a. $\sqrt{121}$

b. $\sqrt{48}$

c. $\sqrt{25x^2}$, $x > 0$

d. $\sqrt{5s^2}$, $s > 0$

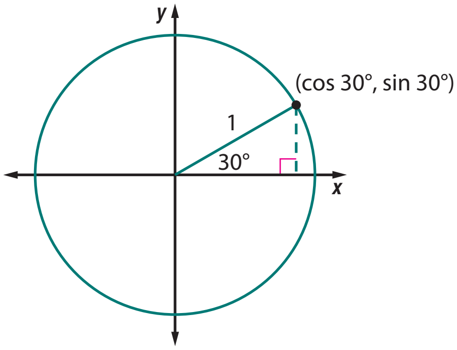
e. $\sqrt{4w^3}$, $w > 0$

f. $\sqrt{\frac{9}{4}x^2}$, $x > 0$

41 Recall that angles can be measured in both degrees and radians, with the two measurement scales related by the fact that $180° = \pi$ radians or $360° = 2\pi$ radians. Complete a copy of the following table showing degree and radian equivalents for some important angles.

Degrees	0		45	60		120		150	180
Radians		$\frac{\pi}{6}$			$\frac{\pi}{2}$		$\frac{3\pi}{4}$		

42 The sketch at the right shows how the cosine and sine of 30° (or $\frac{\pi}{6}$ radians) determine coordinates of a point on the circle of radius 1 centered at the origin.



a. Give exact values of the coordinates.

b. Make a similar sketch showing the location and numerical coordinates of these points on the unit circle.

i. $\left(\cos \frac{3\pi}{4}, \sin \frac{3\pi}{4}\right)$

ii. $\left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right)$

iii. $\left(\cos \frac{3\pi}{2}, \sin \frac{3\pi}{2}\right)$

iv. $\left(\cos \frac{7\pi}{4}, \sin \frac{7\pi}{4}\right)$

v. $\left(\cos \frac{7\pi}{6}, \sin \frac{7\pi}{6}\right)$

43 Without using technology, sketch a graph of each function over the interval $[-2\pi, 2\pi]$. Then give the period and the amplitude of each function. Use technology to check your work.

a. $y = \sin x$

b. $y = 2 \sin x$

c. $y = \sin 2x$

LESSON 3

Inverse Trigonometric Functions

In your previous study, you explored properties of trigonometric functions and used them to model patterns of change in periodic real-world phenomena. For example, the picture above shows Durdle Door in Dorset, England. Because of ocean tides, the water's depth d in feet under the cliff arch is a periodic function of time t in hours after noon.

- On one particular day, the depth function is

$$d(t) = 11 + 3 \sin 0.5t.$$

To find the time(s) between noon and midnight when the depth of the water is 13 feet, you need to solve $11 + 3 \sin 0.5t = 13$.

- On a later date, the depth function is

$$d(t) = 11 - 3 \cos 0.5t$$

To find the time(s) between noon and midnight when the depth of the water is 10 feet, you need to solve $11 - 3 \cos 0.5t = 10$.

Think About This Situation

Solving equations like those that ask for times when tidal water has a specified depth is an inverse problem. You know the output of the function, and you need to find the corresponding input.

a What is the minimum depth of the water on the two particular days? The maximum depth?

b How could you estimate solutions for the given equations by exploring patterns in tables and graphs of function values?

c How would you approach solution of each equation by using algebraic reasoning and properties of the sine and cosine functions?

d What role could the $\sin^{-1}$ or $\cos^{-1}$ calculator commands have in solving these equations?

In this lesson, you will extend your study of trigonometry to consider inverses of the sine, cosine, and tangent functions. The inverse trigonometric functions will allow you to find complete solutions of trigonometric equations like those involved in predicting water depth under the cliff arch at Durdle Door.

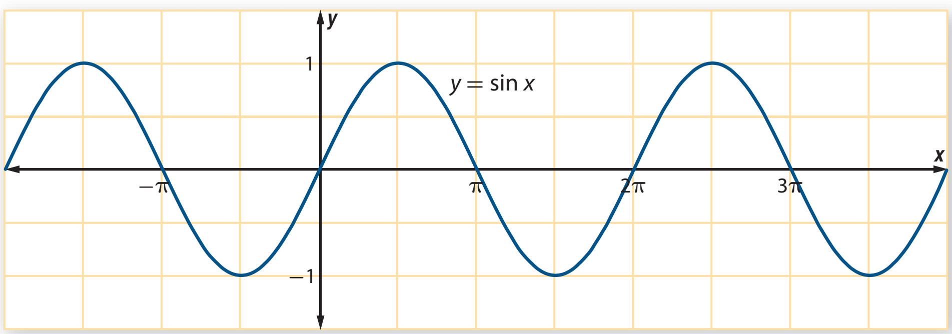
Investigation 1 The Ups and Downs of the Sine

There are many situations exhibiting periodic change that are modeled well by variations of the sine function. Answering questions about those situations often requires solving equations involving that function. As you work on the problems of this investigation, look for answers to these questions:

How is the inverse of the sine function defined?

How can the inverse sine function be used to solve trigonometric equations?

Defining the Inverse Sine Function The next diagram shows a partial graph of the sine function, with scale on the x-axis in radians from -2π to 4π or from about -6.3 to 12.6.



1 Find coordinates of points on the given graph that represent solutions for the following equations. Use what you know about the sine function to find exact values for the coordinates, where possible.

Scan the given graph, a technology-generated table of $y = \sin x$, or a technology-produced graph of $y = \sin x$ to get good estimates of the coordinates as a check on your trigonometric reasoning.

a. $\sin x = 0.5$

b. $\sin x = -1$

c. $\sin x = 1$

d. $\sin x = -0.5$

2 Suppose that you were able to inspect a complete graph of $y = \sin x$.

a. What are the domain and range of $y = \sin x$?

b. For what values of k will the graph show solutions to the equation $\sin x = k$?

c. How many solutions will there be to the equation $\sin x = k$ for each value of k?

d. What do your answers to Parts a–c say about the possibility of defining an inverse for the sine function?

3 Examine the portion of the graph of $y = \sin x$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

a. How many solutions does $\sin x = 0.5$ have on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

b. How many solutions in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ does $\sin x = k$ have for other values of $-1 \leq k \leq 1$?

c. Explain why $y = \sin x$ has an inverse, when x is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

d. The width of the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is π units. Give two other intervals of width π on which $y = \sin x$ has an inverse.

4 Mathematical convention defines the **inverse sine** function as follows.

$$\sin^{-1} k = x \text{ if } \sin x = k \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

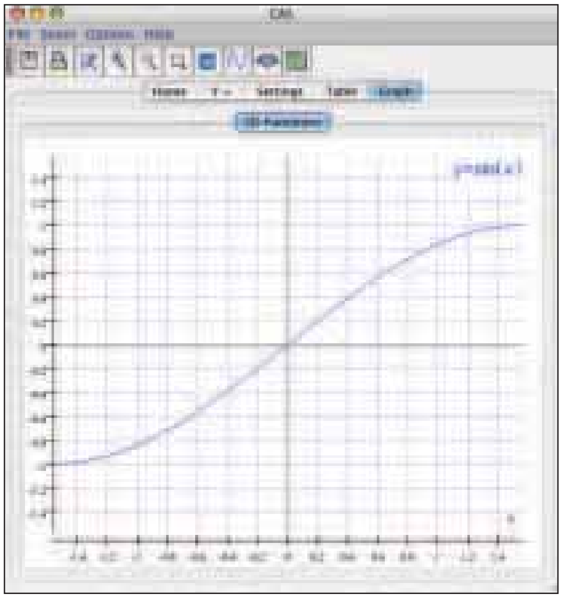
The inverse sine function is also called the *arcsine* function and written arcsin x.

a. Why do you think the definition focuses on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ to define values of the inverse sine function, rather than another interval of length π such as one of those you described in Part d of Problem 3?

b. What are the domain and the range of the inverse sine function?

c. The graph below shows the function $y = \sin x$ on a square window of $-1.57 \leq x \leq 1.57$ and $-1.57 \leq y \leq 1.57$. On a copy of this graph, sketch the graph of $y = \sin^{-1} x$. Check your sketch using technology that displays both functions on a square window.

CPMP-Tools



5 Use algebraic reasoning and **sin^{-1}** on a graphing calculator (2nd SIN) or the inverse sine function of a computer algebra system to find one solution for each of these trigonometric equations.

a. $\sin x = 0.9$

b. $\sin x = -0.8$

c. $5 \sin x + 9 = 12$

d. $11 + 3 \sin 0.5t = 13$

6 The introduction to this lesson proposed two models for predicting the water depth under the cliff arch at Durdle Door in Dorset, England between noon and midnight. One model was

$$d(t) = 11 + 3 \sin 0.5t,$$

where water depth is in feet and time t is in hours after noon. Use that model to analyze the pattern of change in water depth over time. Check your solutions graphically and numerically.

a. What is the depth of the water at 7:00 P.M. on the day that depth tracking begins?

b. What is the maximum depth of the water and at what time will it first occur?

c. At approximately what time(s) between noon and midnight will the water depth be 13 feet?

d. Write an inequality for the time period that the depth is more than 13 feet. Use your solution in Part c to find the solutions of this inequality.

7 Graph $y = \sin^{-1} x$ in a window that shows the full domain and range of the function. Then use the graph to answer these questions.

a. What are the coordinates of the point corresponding to a solution of $\sin x = 0.5$?

b. What are the coordinates of the point corresponding to a solution of $\sin x = -0.8$?

c. What are the x- and y-intercepts of the graph of $y = \sin^{-1} x$?

d. Where is the function $y = \sin^{-1} x$ increasing? Where is it decreasing?

e. Where are the values of $y = \sin^{-1} x$ changing most rapidly? Where are the values changing most slowly?

8 Some consequences of the definition of the inverse sine function are important to emphasize.

a. Consider the expression $\sin(\sin^{-1} k)$.

i. What are the values of the expression for $-1 \le k \le 1$?

ii. What can you say about the expression if $k > 1$ or if $k < -1$?

b. Consider next the expression $\sin^{-1}(\sin x)$.

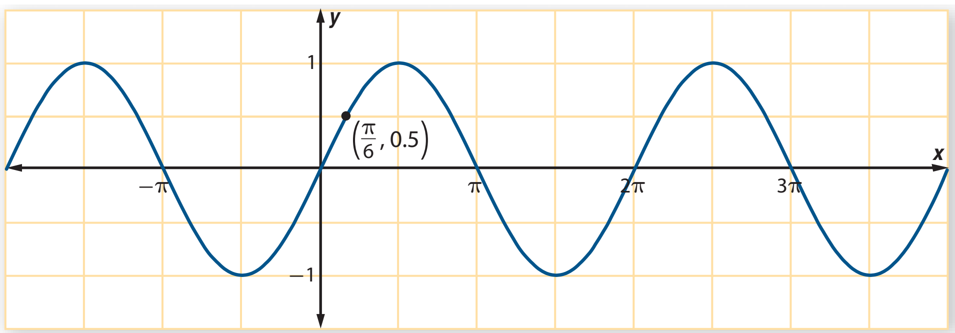
i. What are the values of the expression for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$?

ii. What can you say about the expression if $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$?

Finding All Solutions of Trigonometric Equations Your work on Problems 1 and 2 showed that trigonometric equations like $\sin x = k$ have infinitely many solutions as long as $-1 \le k \le 1$. The inverse sine function identifies one primary solution for each such equation, but there are certainly times when the other solutions are of interest.

For example, if the depth of water under the cliff arch at Durdle Door on one day is given by the function $d(t) = 11 + 3 \sin 0.5t$, the maximum depth of 14 feet occurs at about 3 P.M. But high tides occur at different times on the following days. It would be helpful to be able to use $d(t)$ to predict those other times of maximum water depth.

9 The equation $\sin x = 0.5$ has one solution at $x = \frac{\pi}{6}$ (or $x \approx 0.52$). But studying the graph of $y = \sin x$ reveals other solutions as well.



a. How does the fact that the period of $\sin x$ is 2π make it possible to identify infinitely many solutions of $\sin x = 0.5$ other than $x = \frac{\pi}{6}$? Which of those other solutions are shown on the given graph?

b. The graph of $y = \sin x$ suggests that there are other solutions to the equation $\sin x = 0.5$ that *do not* differ from $x = \frac{\pi}{6}$ by a multiple of 2π.

i. What is the exact radian value of the solution nearest to $x = \frac{\pi}{6}$?

ii. What other solutions shown on the graph differ from that in part i by a multiple of 2π?

c. To find solutions to the equation $\sin x = 0.5$, other than $\frac{\pi}{6}$, students at Providence High School reasoned as follows.

Since the graph of $y = \sin x$ is symmetric about the line $x = \frac{\pi}{2}$, another solution is $\frac{\pi}{3}$ units to the right of $\frac{\pi}{2}$.

Why is the second solution $\frac{\pi}{3}$ units to the right of $\frac{\pi}{2}$? What is that solution?

d. Explain why $\frac{\pi}{6} + 2\pi n$, for any integer n, describes one set of solutions. Write a similar expression representing the remaining solutions.

10 Give formulas that describe *all* solutions to the following equations from Problem 5.

a. $\sin x = 0.9$

b. $\sin x = -0.8$

Inverse Sine of Degrees and Radians In situations where trigonometric functions are used to analyze properties of triangles, the angles are often measured in degrees, not radians. The inverse trigonometric functions can be applied to degree measures as well as radians.

For example, $30° = \frac{\pi}{6}$; so, $\sin 30° = \sin \frac{\pi}{6} = 0.5$. It follows that $\sin^{-1} 0.5 = 30°$, or $\frac{\pi}{6}$ radians.

11 In previous units, you learned the values of the sine function for some special angles. Use what you know about the sine of special angles and the inverse sine function to complete the following table. Use radian measures between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and degree measures between $-90°$ and $90°$.

u	-1	$-\frac{\sqrt{3}}{2}$		$-\frac{1}{2}$	0	$\frac{1}{2}$			1
$\sin^{-1} u$ (in radians)			$-\frac{\pi}{4}$			$\frac{\pi}{6}$		$\frac{\pi}{3}$	
$\sin^{-1} u$ (in degrees)	-90					30	45		

12 Use algebra and trigonometry to find all solutions of the following equations in both degrees and radians. Give exact solutions, if possible.

a. $2 \sin x = -\sqrt{3}$

b. $2 \sin x - 3 = 1$

c. $3 \sin x + 2 = 4$

d. $5 - 3 \sin x = 2$

Summarize the Mathematics

In this investigation, you learned about the inverse sine function and some of its properties. You also used the inverse sine function to solve linear equations involving the sine.

a The sine function defined over the set of real numbers has no inverse function. Why not?

b Explain how the inverse sine function was defined in this lesson in order to overcome the difficulties you noted in Part a.

c Describe the conditions under which $\sin^{-1}(\sin x) = x$. Explain your reasoning.

d Consider an equation of the form $a \sin x + b = c$ which is linear in $\sin x$.

- **i.** How is solving this equation similar to and different from solving $ax + b = c$ which is linear in x?
- **ii.** Under what conditions on a, b, and c does $a \sin x + b = c$ have solutions?
- **iii.** If you know one solution, how many solutions will there be? Describe how to find all solutions.

Be prepared to share your thinking and procedures with the class.

Check Your Understanding

Use your understanding of the inverse sine function to help complete these tasks.

a. Determine whether the following statements are true or false. Explain how you know without use of a calculator or computer inverse sine function.

- **i.** $\sin^{-1} 1 = \frac{\pi}{2}$
- **ii.** $\sin^{-1} 0 = 0$
- **iii.** $\sin^{-1}(-1) = -\frac{\pi}{2}$
- **iv.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$
- **v.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3}$
- **vi.** $\sin^{-1}\left(\frac{1}{2}\right) = 150°$

b. Use algebra and trigonometry to determine whether each equation has a solution for x. If so, find all solutions in radians. Give exact solutions, if possible.

- **i.** $2 \sin x = \sqrt{2}$
- **ii.** $2 \sin x - 2 = 1$
- **iii.** $3 \sin x + 2 = 0$
- **iv.** $2 - 3 \sin x = 2$

Investigation 2 Inverses of the Cosine and Tangent

In Investigation 1, you developed the inverse sine function and used it to find solutions of equations involving the sine. Inverses of the cosine and tangent functions can be used in the same way to answer questions about phenomena that have been modeled with those functions.

As you work on the problems of this investigation, look for answers to these questions:

How are the inverses of the cosine and tangent functions defined?

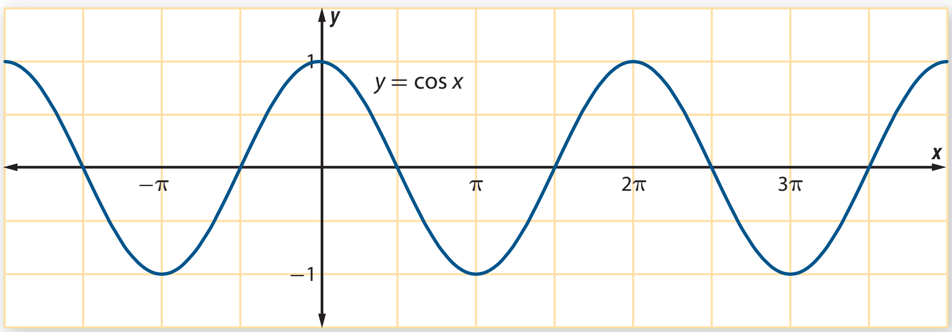
How can the inverse cosine and inverse tangent functions be used to solve equations involving the cosine or tangent?

Defining and Using the Inverse Cosine Function

Suppose that an amusement park is planning a new roller coaster with part of its track shaped like the graph of $y = 12 \cos 0.1x + 6$, where x and y are in meters and $0 \leq x \leq 100$.

For positive values of y, the roller coaster track is above ground. For negative values of y, it is below ground in a tunnel. In order to find the horizontal distance to the point where the roller coaster enters the tunnel, you need to solve $12 \cos 0.1x + 6 = 0$ or its equivalent $\cos 0.1x = -0.5$. An inverse cosine function would help to find the required value of x.

1 Examine the following graph of $y = \cos x$.



a. Why does the cosine function not have an inverse, when defined on its entire domain?

b. Would the cosine function have an inverse if it was defined on these restricted domains?

i. $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ **ii.** $0 \leq x \leq \pi$

iii. $-\pi \leq x \leq 0$ **iv.** $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

c. Of the feasible restricted domains for $y = \cos x$, which seems most useful for defining $\cos^{-1} x$?

The standard mathematical definition of the **inverse cosine** function is

$$\cos^{-1} k = x \text{ if } \cos x = k \text{ and } 0 \leq x \leq \pi.$$

The inverse cosine function is also called the *arccosine* function and written arccos x.

2 Analyze the properties of the inverse cosine function.

a. What are the domain and range of $y = \cos^{-1} x$?

b. What window will show a complete graph of $y = \cos^{-1} x$?

c. Sketch a graph of $y = \cos^{-1} x$. Label the x- and y-intercepts.

d. Label the point on the graph of $y = \cos^{-1} x$ that corresponds to a solution of $\cos x = 0.5$.

e. Label the point on the graph of $y = \cos^{-1} x$ that shows a solution of $\cos x = -0.3$.

3 Use the inverse cosine function, a graph of $\cos x$, and reasoning like what you applied in studying the sine and inverse sine functions to find *all solutions* for these equations.

a. $\cos x = \dfrac{1}{\sqrt{2}}$

b. $6 \cos x = 3$

c. $2 \cos x + \sqrt{3} = 0$

d. $5 \cos 0.3x = 4$

4 The roller coaster described at the start of this investigation had a track matching the graph of $y = 12 \cos 0.1x + 6$, where x and y are in meters and $0 \leq x \leq 100$. For positive values of y, the roller coaster track is above ground. For negative values of y, it is below ground in a tunnel.

a. Graph the function that represents this portion of the track.

b. At what point does the track reach its maximum height?

c. Write and solve an equation that gives the horizontal distance traveled when the roller coaster first enters the tunnel.

d. What are the lowest point(s) of the track? At what value(s) of x does it occur?

e. Write and solve an inequality to find where the roller coaster is in the tunnel.

5 At the start of this lesson, the function $d(t) = 11 - 3 \cos 0.5t$ was proposed as a model of change in tidal water depth at Durdle Door.

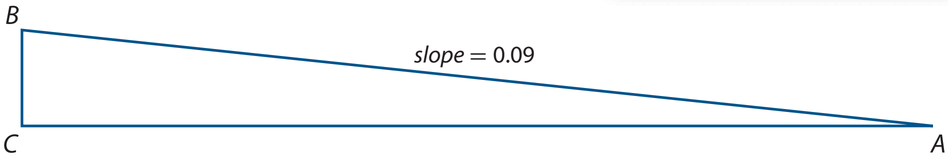
a. If $t = 0$ indicates noon on one day, what is the water depth at that time?

b. Use the inverse cosine to find the first time after noon when the water depth is 10 feet.

c. Use a graph of $d(t) = 11 - 3 \cos 0.5t$ and the inverse cosine function to find *all times* after noon when the water depth is 10 feet.

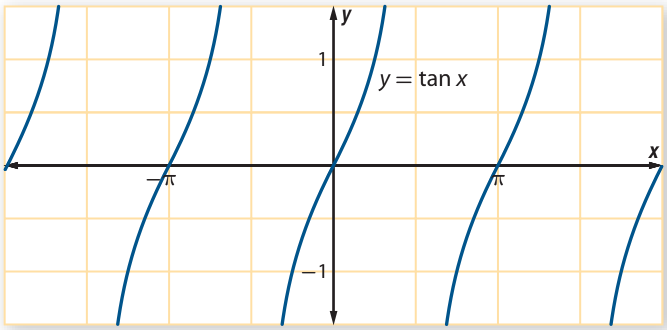
Defining and Using the Inverse Tangent Function If you have ever traveled on a highway with steep hills, you have probably seen signs like the one on the right. It is quite likely that the 9% grade figure was not found by sighting from top to bottom of the hill with a protractor. Instead, road engineers probably used other instruments to measure the horizontal and vertical distances involved.

If the 9% figure represents the slope of the road, it is also the tangent of $\angle A$ in the following diagram.



To find the measure of that angle, you need to solve the equation $\tan x = 0.09$. An inverse tangent function would provide the solution.

6 Examine the following graph of $y = \tan x$.



The graph of $y = \tan x$ is different from the sine and cosine functions in several ways.

a. What are the domain and range of $y = \tan x$?

b. Does $y = \tan x$ have any asymptotes? Explain.

c. What is the period of $y = \tan x$?

d. Why does the tangent function not have an inverse, when defined on its entire domain?

e. Would the tangent function have an inverse if it was defined on these restricted domains? Why or why not?

i. $-\frac{\pi}{2} < x < \frac{\pi}{2}$

ii. $-\pi < x < \pi$

iii. $-\frac{3}{2}\pi < x < 0$

iv. $\frac{\pi}{2} < x < \frac{3\pi}{2}$

f. Of the feasible restricted domains in Part d for $y = \tan x$, which seems most useful for defining $\tan^{-1} x$?

The standard mathematical definition of the **inverse tangent** function is

$$\tan^{-1} k = x \text{ if } \tan x = k \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The inverse tangent function is also called the *arctangent* function and written arctan x.

7 Analyze properties of the inverse tangent function.

a. What are the domain and range of $y = \tan^{-1} x$.

b. Use a calculator or computer function graphing tool in radian mode to help make a sketch of $y = \tan^{-1} x$ in a window like $-5 \leq x \leq 5$ and $-2 \leq y \leq 2$. Then add the following features to your sketch.

i. Sketch the asymptotes.

ii. Label the point on the graph that represents a solution of $\tan x = 1$?

iii. Label the point on the graph that represents a solution of $\tan x = 3.5$.

iv. Label the x- and y-intercepts.

8 With slight adjustments, the strategies that you used to solve equations involving the sine and cosine will work to solve equations involving the tangent. Find all solutions of the following equations in both radians and degrees. Give exact solutions, if possible.

After you find one solution, remember the pattern in the graph of $y = \tan x$ and its period to find all others.

a. $\tan x = 1$

b. $\sqrt{3} \tan x = 1$

c. $-2 \tan x - 3 = 5$

d. $\sqrt{3} \tan x + 4 = -1$

9 To find the angle of depression of a road with 9% grade, you need to solve the equation $\tan x = 0.09$. Find the solution to this equation that makes sense in the problem situation. Give your answer in degrees and in radians.

10 **Properties of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$** In Investigation 1, you derived the following properties of the inverse sine function.

- $\sin(\sin^{-1} k) = k$ provided $-1 \leq k \leq 1$.
- $\sin^{-1}(\sin x) = x$ provided $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- If $-1 \leq k \leq 1$, one solution of $\sin x = k$ is $x = \sin^{-1} k$, and this solution is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (or between $-90°$ and $90°$), inclusive.

a. Replace "sin" with "cos." Make appropriate revisions to state similar properties of the inverse cosine function.

b. Replace "sin" with "tan." Make appropriate revisions to state similar properties of the inverse tangent function.

Summarize the Mathematics

In this investigation, you learned about the inverse cosine and inverse tangent functions, and you used those functions to help solve linear equations involving the cosine or tangent.

a What are the domain and range of $y = \cos^{-1} x$?

b Sketch the graph of $y = \cos^{-1} x$.

c Describe a strategy that uses the inverse cosine function to solve equations in the form $a \cos x + b = c$.

d What are the domain and range of $y = \tan^{-1} x$?

e Sketch the graph of $y = \tan^{-1} x$.

f Describe a strategy that uses the inverse tangent function to solve equations in the form $a \tan x + b = c$.

Be prepared to share your thinking and procedures with the class.

Check Your Understanding

Use your understanding of the inverse cosine and tangent functions to help complete these tasks.

a. Find all solutions of each equation.

i. $3 - 4 \cos x = 6$

ii. $\tan x - 3 = 12$

b. The daily average temperature in a southern hemisphere city like Johannesburg, South Africa, varies throughout the year—warmest from November through February and coolest from May through September.

Suppose that the function $T(m) = 60 + 10 \cos 0.5m$ is a good formula for estimating the daily temperature in degrees Fahrenheit at a time m months into the year. Write and solve equations that answer these questions.

i. At what time of the year is the daily average temperature 65°F?

ii. At what time of the year is the daily average temperature 53°F?

On Your Own

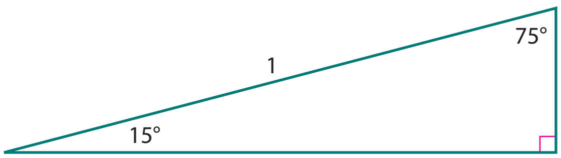
Applications

1 Use the definitions of the sine and inverse sine functions to write the following equations in different equivalent forms.

a. $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ **b.** $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ **c.** $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

2 Use information in the diagram to find the requested values. Be prepared to explain how you know that your answers are correct, without the use of technology.

a. $\sin^{-1} 0.259 =$ ____ **b.** $\sin^{-1} 0.966 =$ ____



3 Use algebraic reasoning and the inverse sine function to find a solution for each of the following equations. Give exact solutions in both radians and degrees, if possible.

a. $\sin x = \frac{1}{2}$ **b.** $\sin 3x = \frac{1}{2}$ **c.** $8 \sin 0.4x - 1 = 3$

4 Use algebraic reasoning and the inverse sine function to find all solutions in radians for the following equations. Give exact solutions, if possible.

a. $2 \sin x + \sqrt{3} = 0$ **b.** $4 \sin x + 3 = 1$ **c.** $3 - \sin x = 2 \sin x$

5 Sarasota, Florida, like many cities in tropical climates, has a seasonal change in population each year. Suppose that the number of people living in Sarasota at any time of the year can be approximated by the function

$$p(t) = 50 + 25 \sin 0.5t,$$

where $p(t)$ is in thousands of people, t is in months after November 1, and $0 \leq t \leq 12$.

a. Graph the population function for a one-year period in an appropriate window. Start your graph using $t = 0$ to stand for November 1.

b. What is the maximum predicted number of people living in Sarasota? When does that maximum occur?

c. What is the minimum predicted number of people living in Sarasota? When does that minimum occur?

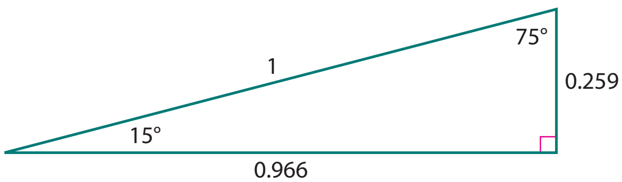
d. On what date(s) in the year is the number of people living in Sarasota about 60,000? Show how to find the answer to this question using algebraic reasoning and the inverse sine function.

6 Determine which of the following mathematical statements are true. Explain how you know without the use of technology.

a. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ **b.** $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

7 Use information in the diagram below to find requested values of the inverse cosine function. Be prepared to explain how you know that your answers are correct, without the use of technology.

a. $\cos^{-1} 0.259 =$ ____ **b.** $\cos^{-1} 0.966 =$ ____



8 Use algebraic reasoning and the inverse cosine function to find a solution for each of the following equations. Give exact solutions in both radians and degrees, if possible.

a. $\cos x = \frac{1}{2}$ **b.** $\cos 5x = \frac{1}{2}$ **c.** $8 \cos 0.4x - 1 = 3$

9 Use algebraic reasoning and the inverse cosine function to find all solutions in radians for the following equations. Give exact solutions, if possible.

a. $2 \cos x - \sqrt{3} = 0$ **b.** $4 \cos x - 3 = 1$ **c.** $2 - \cos x = 3 \cos x$

10 The meandering path of a river often can be approximated by a sine or cosine curve. Suppose that two towns are located so that one is six miles directly east of the other and the path of the river between those towns is approximated by the graph of $y = 2 \cos x$ for $-2 \leq x \leq 4$.

The x-coordinate represents location from west to east in miles and y represents location north or south of the line connecting the two towns in miles.

a. Graph the path of the river over the six-mile interval between the two towns.

b. Answer parts i and ii relative to the origin of this graph.

i. What is the northernmost point of the river between the two towns?

ii. What is the river's southernmost point between the two towns?

c. Suppose a west-east road lies on the line $y = 1.5$ and a north-south road on the line $x = 0$ on the graph.

i. How many bridges are needed for these straight roads?

ii. At what points on the map coordinate system are the bridges located?

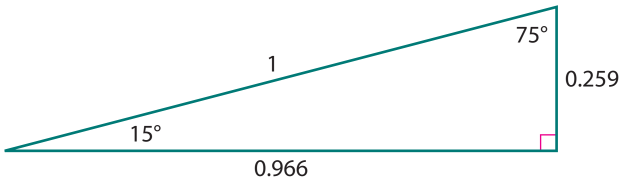
d. If it was decided that the two towns build direct roads connecting the bridge on the north-south road to the bridges on the west-east road, how long will each of those roads be?

11 Determine which of the following mathematical statements are true. Explain how you know, without the use of technology.

a. $\tan 45° = \frac{1}{2}$ **b.** $\tan^{-1} 1 = 45°$ **c.** $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$

12 Use information in the diagram below to find requested values of the inverse tangent function. Explain how you know that your answers are correct, without using the **tan** or **tan⁻¹** function of technology.

a. $\tan^{-1}$ (___) $= 75°$ **b.** $\tan^{-1}$ (___) $= 15°$



13 Use algebraic reasoning and the inverse tangent function to find all solutions for the following equations. Give exact solutions, if possible.

a. $\tan x = 1$ **b.** $5 \tan x = 1$ **c.** $5 \tan x - 1 = 19$

14 The center of the Ferris wheel at a county fair is 7 meters above the ground. The Ferris wheel itself is 12 meters in diameter. The angular velocity of the wheel is 18° per second.

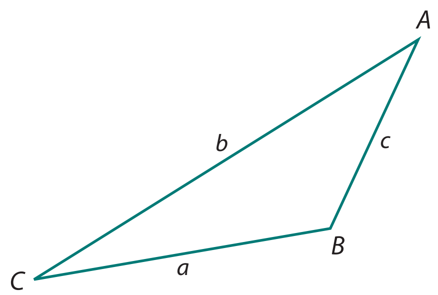
a. Write an equation involving the sine function that gives the distance y in meters above the ground of a seat on the wheel t seconds from the time that it is at the 3:00 position on the wheel. (*Hint:* The seat will return to its starting point in 20 seconds.)

b. Using the equation that you wrote in Part a, write an equation in t whose solutions are the times at which the seat is 10 meters above the ground. Find all solutions between 0 and 30 seconds.

Connections

15 One tool for determining missing parts of a triangle $\triangle ABC$ is the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



a. Write and solve a trigonometric equation whose solution is $m\angle A$ if $a = 2.4$ feet, $b = 3.7$ feet, and $m\angle B = 112°$. Is there just one solution? Explain.

b. Solve the general equation, $\frac{a}{\sin A} = \frac{b}{\sin B}$, for $m\angle A$.

16 Compare the use of inverse functions in solving exponential and trigonometric equations by solving each pair of equations using radians for the sine functions. Note similarities and differences in strategies used.

a. $20(10^x) = 5$ and $20 \sin x = 5$

b. $20(10^{3x}) = 5$ and $20 \sin 3x = 5$

c. $20(10^{3x}) - 3 = 5$ and $20 \sin 3x - 3 = 5$

17 An ice cream store in Indiana is a seasonal business. Stores that remain open all year have a fluctuation in daily income (in thousands of dollars) similar to that shown in the following table.

Date	Daily Income
January 1	0.73
February 1	0.66
March 1	0.85
April 1	1.35
May 1	2.78
June 1	4.67

Date	Daily Income
July 1	5.89
August 1	6.35
September 1	5.77
October 1	4.52
November 1	2.83
December 1	1.44

a. On which of the tabled dates does the store make maximum income? Minimum income?

b. Plot the income data y against x, time in months after January 1. Use January 1 as $x = 0$.

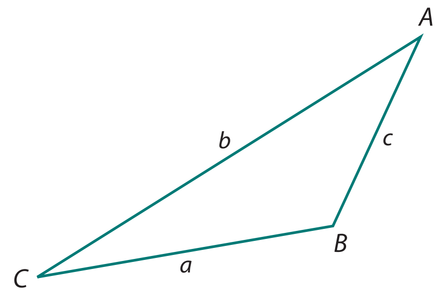
c. Given the shape of the plot in Part b, what kind of function might provide a good fit for these data?

d. Use your calculator or computer software to fit these data to a sine function. On some calculators, this is a statistics function called **sinreg**. What is the equation of the function that best fits the data?

e. Calculate the average daily income of the 12 values in the table. What average daily income is predicted by your equation in Part d?

18 The Law of Cosines can be used to find unknown angle or side measurements in triangles. In $\triangle ABC$,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$



a. Find the measure of $\angle C$ in case $a = 5$, $b = 7$, and $c = 3.25$.

b. Find the measure of $\angle A$ in case $a = 9.6$, $b = 3$, and $c = 11$.

19 The daily attendance of California's scenic Yosemite National Park varies throughout the year with a maximum of about 20,000 visitors on August 1 and a minimum of about 2,000 visitors on February 1.

a. Show that if the function

$$A(m) = 9 \cos 0.5m + 11$$

is used to predict daily attendance in thousands at a time m months after August 1, the function has maximum and minimum values that agree with the given data.

b. About what daily attendance does your equation predict for January 1?

c. On what other day is the predicted daily attendance the same as that of January 1?

d. Members of the Stanford University Hiking Club dislike crowds, but they love to hike in Yosemite. They have decided to avoid the park on days when the predicted attendance is 16,000 or greater. On what days should they avoid Yosemite?

20 Use of the Law of Sines to find side lengths of triangles generally requires a computation like $a = (\sin A)\left(\frac{b}{\sin B}\right)$. Before electronic calculators were invented, that calculation was tedious. It involved multiplying and dividing by decimals accurate to four or more places. To avoid such messy computation, people used logarithms to change the operations to sums or differences that are easier to compute by hand. The logs of trigonometric functions could be found in published tables.

a. Write an equivalent expression for $\log\left((\sin A)\left(\frac{b}{\sin B}\right)\right)$ that requires only addition and subtraction of logarithms. Explain why the result is equal to $\log a$.

b. Suppose $b = 4.5$, $m\angle A = 42°$, and $m\angle B = 103°$.

  i. Use the expression derived in Part a to find $\log a$.

  ii. Use the value of $\log a$ to determine a.

21 The graph of each equation in the form $y = kx$, $x \geq 0$ meets the positive x-axis at the origin to form an angle with measure between $-90°$ and $90°$. The graph of the equation $y = -2x$, $x \geq 0$ meets the positive x-axis at an angle of approximately $-63°$.

a. Complete a copy of the following table to explore the relationship between angle and slope.

Slope	−4	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	4
Angle		−63									

b. Plot the (*slope, angle*) data.

c. Identify a function type that seems likely to model well the dependence of *angle measurement* on *slope*. Use a curve-fitting tool to find the best-fit model of that type.

d. Identify the intercepts, local maximum and minimum points, and asymptotes of the function model from Part c.

22 Use the coordinate definition of the trigonometric functions and properties of their inverses to determine the values of these expressions.

a. $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

b. $\tan\left(\arcsin\left(-\frac{3}{4}\right)\right)$

c. $\sin\left(\cos^{-1}\left(-\frac{1}{5}\right)\right)$

d. $\sin^{-1}(\sin(-300°))$

e. $\arccos\left(\sin\left(-\frac{\pi}{4}\right)\right)$

f. $\tan(\cos^{-1} x)$, $0 < x < 1$

23 Use your calculator to complete a copy of the following table. Explain interesting patterns you see in the table. Explain your results.

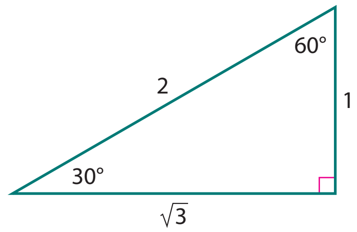
	Radian Measure (x)						
	1	2	3	4	5	6	7
$\sin^{-1}(\sin x)$	1						
$\cos^{-1}(\cos x)$				2.2832			
$\tan^{-1}(\tan x)$							0.7168

Reflections

24 Sketch the graph $y = \sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and the line $y = x$ on the same coordinate system. Sketch the image of the graph of $y = \sin x$ reflected across the line $y = x$. What is the equation for this reflection image? Explain.

25 To remember the meaning of the inverse sine function, many students say to themselves:

The notation $\sin^{-1} x$ means to find the angle whose sine is x. For example, $\sin^{-1} 0.5 = 30°$, or $\frac{\pi}{6}$, because in a 30°-60° right triangle, the side opposite the 30° angle is one-half the hypotenuse.



a. Do you agree that this is a helpful and correct way to think about the inverse sine function?

b. What does this way of thinking and the given drawing suggest is the value of $\sin^{-1} \frac{\sqrt{3}}{2}$?

c. What happens when you try to apply this way of thinking to find $\sin^{-1} (-0.5)$?

d. What somewhat similar way of thinking about inverse functions can be used when you are asked to find $\log x$, for example, to find $\log 1{,}000$ or $\log 0.01$?

26 Explain the meaning of each of the following expressions. Evaluate them for $t = 0.5$.

a. $\cos^{-1} t$

b. $\cos (t^{-1})$

c. $(\cos t)^{-1}$

d. $\arccos t$

27 Each of the following equations is true for some values of t and false for other values of t.

- For each equation, give one value of t for which the equation is true and one value for which it is false.
- Describe the set of all t for which each equation is true.

a. $\sin^{-1} (\sin t) = t$ (radians)

b. $\cos^{-1} (\cos t) = t$ (degrees)

c. $\tan^{-1} (\tan t) = t$ (radians)

Extensions

28 Simple sound waves can be represented by functions of the form $y = a \sin bx$. Certain sound waves, when played simultaneously, will produce a particularly pleasant sensation to the human ear. Such sound waves form the basis of music. For example, two sound waves are said to be separated by an *octave* if the ratio of their frequencies is 2:1.

a. The frequency of a middle C note is about 264 Hz, or 264 cycles per second. What is the period of this sound wave in fractions of a second?

b. If the note has amplitude 70, how can its fluctuating air pressure y be written as a function of time x in seconds?

c. Suppose that second note with equal amplitude and higher frequency is separated by an octave from middle C. Write a function for the sound wave associated with this note.

d. To begin to make music, these two notes can be played together. Write and graph a third function that is the sum of these two for $0 \leq x \leq 0.01$.

e. Suppose another note is separated from middle C by a third. That is, the frequencies of the two notes are in the ratio of 5:4. What are the two possible frequencies of this note?

29 Verify that each equation is true for all x where $0 \leq x \leq 1$. You will be showing that each equation is an *identity*, that is, a statement that is true for all replacements of the variable for which the statement is defined. (*Hint:* In Part a, the rotational symmetry of the sine function about the origin means that $\sin(-x) = -\sin x$ for all x.)

a. $\sin^{-1}(-x) = -\sin^{-1} x$

(*Hint:* In Parts b–d, make use of the fact that a point on the terminal side of $\sin^{-1} x$ at distance 1 from the origin has coordinates $\left(\sqrt{1 - x^2}, x\right)$ and a point on the terminal side of $\cos^{-1} x$ at distance 1 from the origin has coordinates $\left(x, \sqrt{1 - x^2}\right)$.)

b. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

c. $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

d. $\tan(\cos^{-1} x) = \frac{\sqrt{1 - x^2}}{x}$

30 More complicated trigonometric equations and other equations containing both trigonometric functions and other functions can be solved using symbolic reasoning. Use symbolic reasoning to solve the following equations in radians. *Hint:* It might help if you start by thinking about $\cos x$, $\sin x$, and $\tan x$ as unknown quantities u and solving first for u.

a. $\cos^2 x + 2\cos x + 1 = 0$

b. $2\sin x \cos x + \sin x = 0$

c. $\tan^2 3x = 4\tan 3x$

d. $\sin^2 2x = 5\sin 2x - 6$

e. $\log(\cos x) + 2.5 = 1.9$

f. $5(10^{\tan x}) + 15 = 64$

31 At one time in the history of scientific computer programming, some languages, like BASIC and FORTRAN, had only one inverse trigonometric function: the inverse tangent. The inverse sine and inverse cosine were both computed in these languages using expressions that involved the inverse tangent. Complete the following tasks to see how this can be done.

a. Let $y = \sin^{-1} x$. The goal is to write y in terms of the inverse tangent of a function of x. Start by writing x in terms of y.

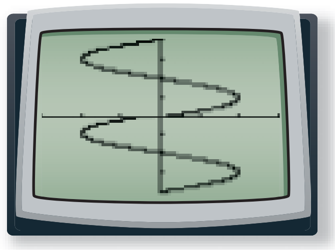
b. Consider the angle with measure y in standard position. Use your equation for x in Part a to write an expression for $\tan y$ in terms of x.

c. Transform this equation to one in which y is written in terms of the inverse tangent of a function of x.

d. Show that your equation gives correct values of $\sin^{-1} 0.5$ and $\sin^{-1}(-0.8)$.

e. Repeat the steps in Parts a–d to show how to write $\cos^{-1} x$ in terms of the inverse tangent of a function of x. You will need to take care that the equation you arrive at gives the correct value of $\cos^{-1} x$ if $x < 0$.

32 The graph of $x = \sin y$ is given at the right. This graph is clearly not the graph of a function of x. For that reason, your calculator in function mode will not draw this graph. Rather, you will need to set your calculator to parametric mode. With parametric equations, a third variable t is introduced, and the x- and y-coordinates are written as separate functions of t.

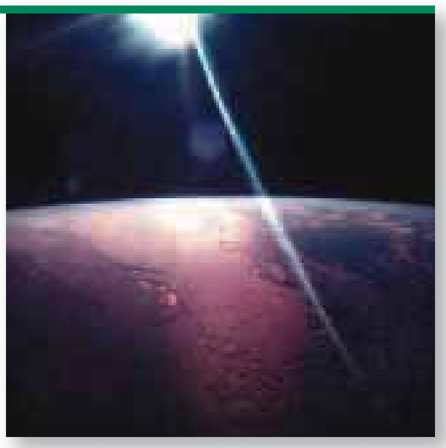


a. With your calculator set to parametric mode, press Y=. You will then enter a function of t for X_T (or X_{1T}). Enter **sin(T)**. If the resulting pair of equations, X_T and Y_T, is to be equivalent to $x = \sin y$, what should you enter for Y_T?

b. Experiment with window settings to get a graph that looks like this one. A good choice for Tstep is 0.1. What settings for x, y, and t worked for you?

c. Without changing the window settings for x and y, how could you change the settings for t so that the resulting graph is that of $y = \sin^{-1} x$? Explain.

d. Repeat Parts a–c with the graph of $x = \cos y$.



33 Earth travels around the Sun in an elliptical orbit. The distance d in kilometers between Earth and the Sun at time t in days since January 1 is approximated by the following equation.

$$d = \frac{1.5 \times 10^8}{1 - 0.0166 \cos\left(\frac{2\pi(t - 185)}{365}\right)}$$

a. What is the minimum distance from Earth to the Sun? On what date does the minimum distance occur?

b. What is the maximum distance from Earth to the Sun? On what date does the maximum distance occur?

c. Write an equation whose solutions are the two days during the year at which Earth is 1.48×10^8 km from the Sun. Use symbol manipulation methods to solve the equation. Check your solutions by substitution.

34 The center of a water wheel with radius 4 meters is 2.5 meters above the surface of the water. A point P is located at (4, 0) on the circumference of the wheel when the wheel begins to rotate counterclockwise through an angle θ in radians.

a. Explain why point P is below the water's surface between consecutive solutions of $4 \sin \theta + 2.5 = 0$.

b. Find the interval of θ between 0 and 2π for which point P is below the water's surface by solving this equation.

35 Use symbolic reasoning and properties of the inverse trigonometric functions (in radian mode) to solve the following equations for x.

a. $2 \sin^{-1} x = 1$

b. $\cos^{-1} 2x = 0.3$

c. $3 \tan^{-1} \pi x = -3$

Review

36 Solve each equation or inequality. Graph the solution to each inequality on a number line.

a. $4 + 2x(x - 5) = (x + 3)(2x - 8)$

b. $2x^2 + 6x = 8$

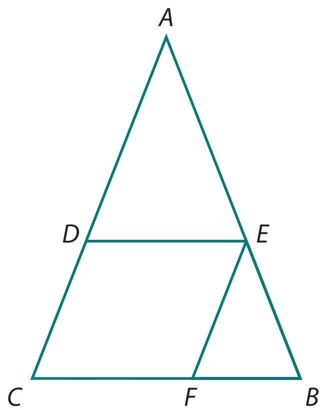
c. $x + 3 = \frac{1}{x}$

d. $(x^2 + 4x)(x - 3) = 0$

e. $10x - 21 \leq x^2$

f. $12 - 8x \geq 4(x + 9)$

37 In the diagram at the right, $AC = AB$, $AD = AE$, and $DC = EF$.



a. Prove that $\triangle EBF$ is isosceles.

b. Prove that quadrilateral $DEFC$ is a parallelogram.

c. Is $\triangle EBF \sim \triangle ABC$? Provide reasoning to support your answer.

d. If $m\angle C = 80°$, find the following angle measures.

i. $m\angle B$ **ii.** $m\angle ADE$

iii. $m\angle A$ **iv.** $m\angle EFC$

38 Antonio has an account balance of \$1,500 on his credit card. His credit card company charges a monthly interest rate of 1.5%. He can afford to pay \$50 a month to pay off his debt and has made a pledge to himself not to purchase anything more with his credit card.

a. How much will his credit card balance be after 1 month? After 2 months?

b. Write a recursive formula for the account balance from one month to the next.

c. How long will it take him to reach a zero balance?

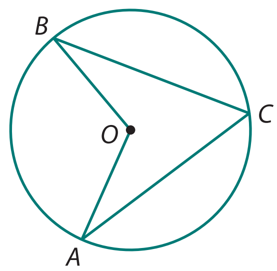
d. How much interest will he have paid when his account reaches a zero balance?

39 Use relationships between angle measures and arc lengths to determine each indicated value.

a. In the circle with center O, $m\angle BOA = 5x + 16°$ and $m\angle BCA = 3x - 2°$.

i. $m\angle BCA$

ii. $m\widehat{BCA}$



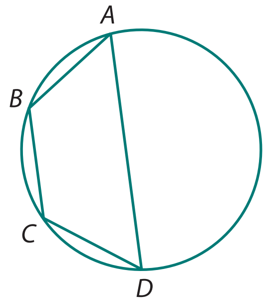
b. In the circle at the right, $AB = BC = CD$ and $m\angle C = 125°$.

i. $m\widehat{BAD}$

ii. $m\widehat{BCD}$

iii. $m\widehat{BC}$

iv. $m\angle BAD$

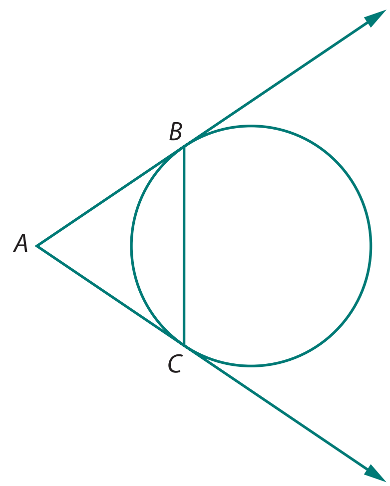


c. In the diagram at the right, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangent to the circle at points B and C, $m\angle A = 68°$, and $AC = 8$ in.

i. AB

ii. BC

iii. Radius of the circle



40 For each rational function below, use algebraic reasoning to:

- find coordinates of all x-intercepts and the y-intercept of the graph.
- describe the domain of the function.
- find equations of all vertical and horizontal asymptotes.
- sketch a graph on which you label intercepts and asymptotes.

a. $f(x) = \dfrac{5}{x - 2}$

b. $g(x) = \dfrac{3x - 4}{2x + 1}$

41 A national survey of 13–17-year-olds indicated that 43% of teens say they have experienced some form of cyberbullying. Suppose you randomly select 120 teens to survey about cyberbullying.

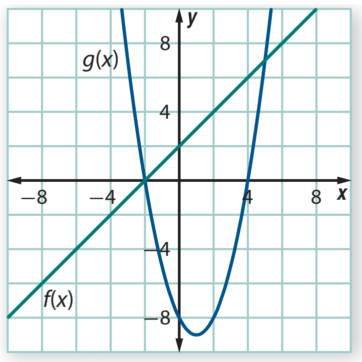
a. How many of them would you expect to say they have experienced some form of cyberbullying?

b. Is the binomial distribution of the number of teens who say they have experienced cyberbullying approximately normal?

c. What is the standard deviation of this binomial distribution?

d. Would you be surprised to find that only 42 teens in your sample reported having been cyberbullied? Explain.

42 Consider the graphs of $f(x)$ and $g(x)$ shown below. The two graphs intersect at the points $(-2, 0)$ and $(5, 7)$.



a. For what values of x is $f(x) - g(x) < 0$?

b. Find a formula for $f(x)$.

c. If $g(x)$ is a quadratic function with vertex $(1, -9)$, write the vertex and standard expressions for $g(x)$.

d. What is the degree of each of the following? Explain your reasoning.

- $f(x) \cdot g(x)$
- $f(x) + g(x)$

LESSON 4

Looking Back

The lessons of this unit extended your knowledge and skill in work with algebraic and trigonometric functions and expressions in three ways. First, you learned what it means for a function to have an inverse, how to find inverse functions when they exist, and how to use inverse functions to solve a variety of problems. Second, you learned the definition and key properties of an important inverse function that gives base-10 logarithms, and you developed strategies for using that function to solve exponential equations. Third, you extended your understanding of the basic sine, cosine, and tangent functions. Then you learned the definitions, properties, and applications of three inverse trigonometric functions.

The next several tasks give you an opportunity to review your knowledge of inverse functions and apply that knowledge to some new contexts.

1 In which of these situations will the indicated function have an inverse? For those functions that do have inverses, explain what information the inverse would provide.

a. In most states, there is a rule that shows how to calculate sales tax as a function of the price of any purchase.

b. The price for a package of cheese sold in a market is a function of the weight of the package.

c. The time that it takes the sound of thunder to reach the ears of a person is a function of the distance from the lightning strike producing the thunder to the listener's ears.

d. The depth of water at a particular location in an ocean harbor varies over time.

2 Sketch graphs of these functions. Tell which have inverses. Then find rules and sketch graphs for those that do.

a. $f(x) = 0.5x$

b. $g(x) = 1.5x + 4$

c. $r(t) = \frac{500}{t}$

d. $s(x) = \log x$

3 Light, radio and television signals, x-rays, and the waves produced by alternating electric current are forms of electromagnetic radiation. The next table gives frequencies for a sample of such electromagnetic waves.

a. Write each frequency in scientific notation.

b. Suppose that you had a table that gave logarithms only for numbers between 0 and 10. Show how you could use those table values, the results of your work in Part a, and properties of logarithms to estimate the logarithms of the given frequencies.

Type of Electromagnetic Wave	Frequency in Cycles per Second
Alternating electrical current	60
AM radio	from 540,000 to 1,600,000
Television and FM radio	from 54,000,000 to 216,000,000
Light	from infrared 390,000,000,000,000 to ultraviolet 770,000,000,000,000
x-rays	from 30,000,000,000,000,000 to 100,000,000,000,000,000,000

4 If you buy a new car for $25,000, its resale value decreases by about 20% each year. Write and solve equations representing these questions.

a. When will the value of the car be only $10,000?

b. How long will it take for the value of the car to be reduced to only $5,000?

c. When is the car worth half the purchase price?

5 In wildlife preserves, like national parks, the populations of many animal species vary periodically in relationship to the populations of other species that are either predators or prey. For example, rabbits are prey for foxes, so when the fox population is high, it causes the rabbit population to fall, and vice versa.

a. Suppose that the fox population in a study region is given by the function $f(t) = 20 \sin t + 100$, where t is the time in years after the first census in the region.

i. What are the period and amplitude of cycles in the fox population?

ii. At what time(s) was the fox population a maximum and when a minimum during the first cycle of change?

iii. At what time(s) will the fox population be about 115? When about 90?

b. Suppose that the rabbit population in the same study region is given by the function $r(t) = 300 \cos t + 2{,}500$.

i. What are the period and amplitude of cycles in the rabbit population?

ii. At what time(s) was the rabbit population a maximum and when a minimum during the first cycle of change?

iii. At what time(s) will the rabbit population be about 2,750? When about 2,300?

Summarize the Mathematics

In this unit, you explored inverses of functions with particular attention to inverses of exponential functions and of trigonometric functions.

a What does it mean to say that two functions f and g are inverses of each other?

b How can you determine whether a function $f(x)$ has an inverse by studying:

i. a table of $(x, f(x))$ values?

ii. a graph of the function?

iii. a definition in words or a symbolic rule for the function?

c Among the families of functions that you have studied most closely and used most often:

i. which almost always have inverses?

ii. which do not generally have inverses?

d How can the statement "$\log_{10} a = b$" be expressed in equivalent form using exponents?

e How can properties of logarithms be used to write these algebraic expressions in equivalent form?

i. $\log mn = \ldots$ **ii.** $\log m^n = \ldots$ **iii.** $\log \frac{m}{n} = \ldots$

f How can properties of logarithms be used to solve exponential functions like $a(b^x) = c$?

g What does it mean to say that $\sin^{-1} x = k$?

h What are the domain and range of $\sin^{-1} x$? Of $\cos^{-1} x$? Of $\tan^{-1} x$?

i How can inverse trigonometric functions be used to solve trigonometric equations?

Be prepared to share your examples and descriptions with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.